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Mathematical Reviews

Edited by

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J. V. Wehausen, *Executive Editor*

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Mathematical Reviews

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HISTORY

Bruins, E. M. On Babylonian geometry. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 16–23 (1955).

Mau, Jürgen. Zum Problem des Infinitesimalen bei den antiken Atomisten. Deutsche Akad. Wiss. Berlin. Inst. Hellen.-Röm. Philos. Veröff. Nr. 4, 48 pp. (1954).

Baranovskaya, L. S. The first work on mathematics in the Mongolian language. Trudy Inst. Istor. Estest. Tehn. 1, 53–84 (1954). (Russian)

Clagett, Marshall. A medieval Latin translation of a short Arabic tract on the hyperbola. Osiris 11, 359–385 (1954).

***Taylor, E. G. R. The mathematical practitioners of Tudor & Stuart England.** Cambridge, at the University Press, 1954. xi+443 pp. (12 plates+2 inserts). $9.50.

The description given on the cover well describes this volume. It is a chronicle of those lesser men—teachers, text-book writers, instrument makers, navigators, land-surveyors, map-makers—during the period 1485–1714 when English science made its first significant and swift advance. Great or familiar names—Hariot, Dee, Mercator, Wren, Hooke, Tompion, Newton—occur, but the story as a whole is concerned with the more humble but no less devoted men whose work created the climate in which advances were possible. In a Foreword, the Astronomer Royal, Sir Harold Spencer Jones F.R.S., has remarked: "Professor Taylor has given a fascinating account of these mathematical practitioners and of the development of ideas, methods and instruments from Tudor times to early in the eighteenth century".

The book falls into three parts: (1) The Narrative, very happily written, which runs chronologically, pp. 1–162; (2) The Practitioners, with Biographical Notes, pp. 165–307; and (3) Works on the Mathematical Arts and Practices, with Descriptive Notes, pp. 311–431. These parts include nearly 600 practitioners and over 600 works; and they are followed by a bibliography and an index, which is confined however to the names mentioned in parts (2) and (3) only.

The book is a mine of interesting information and a most useful work of reference: one only regrets that the index does not cover part (1) also, and that it omits the subjects and the instruments. The work, which is excellently produced, includes a dozen plates, and other illustrations.

H. W. Turnbull (Millom).

***Dugas, René. La mécanique au XVII^e siècle. (Des antécédents scolastiques à la pensée classique.)** Editions du Griffon, Neuchâtel, 1954. 620 pp.

After writing an excellent general Histoire de la mécanique [Griffon, Neuchâtel, 1950; MR 14, 341], the author has set himself in the present volume a task of a rather different sort. Selecting the earlier of the two greatest centuries of mechanics, he has attempted to show how not only the science of mechanics but also the mechanistic views of science and philosophy developed and influenced one another at that time. Since relatively little concerns the mathematical solution of problems in mechanics—in the author's term, "positive" mechanics—this excellent book will not be reviewed in detail here.

The author has followed the method of his earlier treatise in giving, for the most part, extended quotations of the original writings followed by brief but often very necessary explanations. In this work, the greater space available has made this plan even more successful than before, and occasional personal details add not only to its interest but also to our consequence in the development. The author's emphasis of the philosophical side of mechanics at this period is necessary and enables the reader to understand many details and several general trends which otherwise must seem incomprehensible.

While the author makes it clear that he does not personally adhere to any of the current parties in regard to the history of science, his work is colored by their existence. Thus while he gives less than thirty pages to Galilei, he devotes more than eighty to Descartes and at least thirty more to the Cartesians. For the first time in a work intended for the general scientific public appears an explanation of what Descartes really gave to mechanics and physics: hardly a single detail that has retained value, but rather the first claim that nature is a machine subject to rational laws and the first attempt, brilliant if crude and mainly false, to embrace all these laws in a single system.

While the author has taken pains to set the seventeenth century in relation to its antecedents, he has not attempted to trace within it the origins of the more formal eighteenth century or its positivist and materialist successor. Thus, for example, his eighty page treatment of Newton emphasizes Newton's struggles with his contemporaries and the aspects of his work which were foremost in these struggles, but one gets only half a view of Newton's importance, since those aspects of his work which were to dominate much of science in the eighteenth century are barely noticed.

Those interested in the foundations of mechanics or in the history of science will find extraordinary value in every part. The book is beautifully written, and it is hard to put it down or skip a word.

C. Truesdell.

Rényi, Alfréd. A short review of the history of probability calculus. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 447–466 (1954). (Hungarian)

Le Lionnais, F. La contribution de la Grande Bretagne au développement des sciences mathématiques depuis un siècle. Osiris 11, 40–49 (1954).

Belozerov, S. E. Mathematics in the Rostov University. Istor.-Mat. Issled. 6, 247–352 (1953). (Russian)

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Natucci, A. L'origine dell'algebra. Giorn. Mat. Battaglini (5) **2**(**82**), 429–436 (1954).

Itard, Jean. Sur la méthode de Tannery en histoire des mathématiques. Rev. Hist. Sci. Appl. **7** (1954), 326–332 (1955).

Tenca, Luigi. Relazioni fra Stefano Angeli e Vincenzio Viviani. Ist. Veneto Sci. Lett. Arti. Atti **112**, 1–15 (1954).

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Bašmakova, I. G. Differential methods in the works of Archimedes. Istor.-Mat. Issled. **6**, 609–658 (1953). (Russian)

Clagett, Marshall. The De curvis superficiebus Archimenidis: a medieval commentary of Johannes de Tinemue on book I of the "De sphaera et cylindro" of Archimedes. Osiris **11**, 295–358 (1954).

Whittaker, Edmund T. Ernest William Barnes, 1874–1953. Obit. Notices Roy. Soc. London **9**, 15–25 (1 plate) (1954).

Gnedenko, B. V. On a work of P. L. Čebyšev not occurring in the complete collected works. Istor.-Mat. Issled. **6**, 215–222 (1953). (Russian)

Dahiya, S. A. P. L. Čebyšev and the popularization of mathematics in Russia. Istor.-Mat. Issled. **6**, 239–244 (1953). (Russian)

Tenca, Luigi. Otto lettere inedite dell'abate Antonio Conti. Ist. Veneto Sci. Lett. Arti. Atti **112**, 103–119 (1954).

Turnbull, H. W. Obituary: Cuthbert Edmund Cullis. J. London Math. Soc. **30**, 252–255 (1955).

Tuman'yan, T. G. Euclid's "Elements" in old Armenian sources. Istor.-Mat. Issled. **6**, 659–671 (1953). (Russian)

Kuznecov, B. G. The problem of the true motion of the earth in Galileo's "Dialogue". Trudy Inst. Istor. Estest. Tehn. **1**, 249–266 (1954). (Russian)

Badillo, M. C. Obituary: R. H. Germay. Gaceta Mat. (1) **6**, 147–151 (1 plate) (1954). (Spanish)

Yuškevič, A. P. The arithmetical treatise of Muhammad ibn Mûsâ al Khwârizmi. Trudy Inst. Istor. Estest. Tehn. **1**, 85–127 (1954). (Russian)

Kirik Novgorodec. Instruction on how one may learn to reckon the years. Istor.-Mat. Issled. **6**, 173–191 (1953). (Russian)

Translation of an Old Slavonic text from the 12th century. A facsimile of the original faces the translation.

Zubov, V. P. Notes to "Instruction on how one may learn to reckon the years" by Kirik Novgorodec. Istor.-Mat. Issled. **6**, 192–195 (1953). (Russian)

Commentary on the work listed above.

Zubov, V. P. Kirik Novgorodec and the old Russian divisions of the hour. Istor.-Mat. Issled. **6**, 196–212 (1953). (Russian)

Budylina, M. V. An unpublished letter of Leibniz. Trudy Inst. Istor. Estest. Tehn. **1**, 309–316 (1954). (Russian)

Zubov, V. P. Leonardo da Vinci and the work of Vitelo, "Perspectiva". Trudy Inst. Istor. Estest. Tehn. **1**, 219–248 (1954). (Russian)

Rogačenko, V. F. On the discovery by N. I. Lobačevskiĭ of a method of approximate solution of numerical algebraic equations. Istor.-Mat. Issled. **6**, 477–494 (1953). (Russian)

Mandryka, A. P. N. V. Maievskiĭ—the founder of contemporary exterior ballistics. Trudy Inst. Istor. Estest. Tehn. **1**, 146–192 (1954). (Russian)

Otradnyh, F. P. An episode from the life of academician A. A. Markov. Istor.-Mat. Issled. **6**, 495–508 (1953). (Russian)

Taton, René. Un texte inédit de Monge: Réflexions sur les équations aux différences partielles. Osiris **9**, 44–61 (1950).

Makarova, V. I. A forgotten portrait of M. V. Ostrogradskiĭ. Trudy Inst. Istor. Estest. Tehn. **1**, 317–319 (1954). (Russian)

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Obituary: Dimitrie Pompeiu. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 421–422 (1954). (Romanian)

Natucci, A. Nel primo centenario della nascita di Gregorio Ricci-Curbastro. Giorn. Mat. Battaglini (5) **2**(**82**), 437–442 (1954).

Gold, E. Lewis Fry Richardson, 1881–1953. Obit. Notices Roy. Soc. London **9**, 217–235 (1 plate) (1954).

Levey, Martin. Abraham Savasorda and his algorism: a study in early European logistic. Osiris **11**, 50–64 (1954).

Depman, I. Ya. V. A. Steklov at Petersburg University. Istor.-Mat. Issled. **6**, 509–528 (1953). (Russian)

Bahmutskaya, Ê. Ya. On the pedagogical activity of V. A. Steklov in the Kharkhov Technological Institute. Istor.-Mat. Issled. **6**, 529–534 (1953). (Russian)

Kasumhanov, F. A. The theory of continuous quantities and the study of number in the works of Muhammad Nasir-èddin Tusi. Trudy Inst. Istor. Estest. Tehn. **1**, 128–145 (1954). (Russian)

Krall, Giulio. Vito Volterra. La matematica e la scienza del suo tempo. Civiltà delle Macchine **3**, no. 1, 64–77 (1955).

FOUNDATIONS

Have

★**Leblanc, Hugues. An introduction to deductive logic.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1955. xii+244 pp. $4.75.

This book contains all the material which is usually covered in a first course in logic for sufficiently mature students: an introduction on semiotics, sentential and quantificational logic and their formalization (both in axiomatic form and as a system of natural deduction), the logic of identity, classes and relations (an algebraic treatment of classes and relations is included, but its peculiar character is not sufficiently stressed), and, finally, an introduction to metamathematics, which deals with the consistency, completeness, and decidability of the afore-mentioned systems. The treatment of this material satisfies high demands as to lucidity and rigor, but it seems to be somewhat too "heavy" for independent reading. In the hands of a competent teacher, the work will prove a reliable textbook. It seems a pity that the author has given an outline of Gödel's original completeness proof rather than a full treatment of one of the simplified arguments now available; but this is a matter of taste. A serious gap is created by the absence of any application to concrete deductive theories. There is an excellent collection of exercises and a useful Selected Bibliography. *E. W. Beth* (Amsterdam).

Have

★**Rosser, J. Barkley. Deux esquisses de logique.** Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1955. 69 pp. 900 francs.

These "Sketches" are based on two short series of lectures which were given by the author in Paris early in 1954. They are chiefly of an expository character and describe classical results as well as more recent work including some of the author's own contributions.

The first sketch deals with combinatorial logic and its connection with general recursive functions and with the theory of lambda-conversion. Having explained the type of problem with which combinatorial logic is concerned, the author proceeds to prove Schönfinkel's result that all combinators can be obtained by the concentration of S and K. Then follows the axiomatic theory of combinatorial logic in its two versions, due to Curry and Rosser respectively. Next comes a short introduction to the theory of lambda-conversion of A. Church and the proof that it is equivalent to the weak axiomatic theory of combinatorial logic. The connection between lambda-conversion and general recursive functions is discussed in detail. The amount of material covered with apparent ease in a relatively small space is impressive.

The second sketch is concerned with the formulations of the theory of models. The completeness theorem of the lower predicate calculus is derived but the deductive aspects of the proof are not considered in detail. The only atomic relation submitted is the epsilon of set theory but, as is well-known, this restriction is only apparent. The completeness theorem is here credited to Löwenheim and Skolem. However, while some of the essential ideas of Gödel's paper on the subject were in part anticipated by Löwenheim, it appears to the reviewer that the credit for having specifically established the completeness of the deductive theory of the lower predicate calculus should still go to Gödel.

Having added classes to the original sets of the language, the author next considers the more subtle problems which are associated with the formalisation of model-theory, more particularly of the above-mentioned completeness theorem. The existence of non-normal models is discussed (implying, as usual, an intuitive understanding of the meaning of normality). The final chapter deals with certain questions of independence in axiomatic set theory. *A. Robinson.*

Harrop, R. An investigation of the propositional calculus used in a particular system of logic. Proc. Cambridge Philos. Soc. **50**, 495–512 (1954).

Let A be the propositional part of Ackermann's logic [J. Symb. Logic **15**, 33–57 (1950); MR **12**, 384]. It is shown here that this system lacks some of the most useful properties of equivalence, where by the equivalence of two formulas X and Y we mean that substitution of one for the other in provable formulas leads again to provable formulas. If $X\rightarrow Y$, $Y\rightarrow X$, $\sim X.\rightarrow\sim Y$, $\sim Y.\rightarrow\sim X$ are all provable, it does not follow that X and Y are equivalent. In fact, the author proves that no two different formulae are equivalent. (It is however not clear what is meant by 'different'. The theorem seems to fail if X is $U \& V$ and Y is $V \& U$.) In view of this and also in order to have a form of the deduction theorem hold in A, or in an extension of A, two other systems A' and A'' are introduced, both stronger than A. A'' was also introduced by Ackermann in a later paper [Math. Z. **55**, 364–384 (1952); MR **14**, 344]. A'' is a proper extension of A'. Various properties of A, A', A'' are examined. The author mentions that there is an effective method of deciding whether any given formula of A is a theorem of A or not, and conjectures that there is a corresponding procedure for A' and A''. *I. N. Gál* (Ithaca, N. Y.).

Wang, Hao. The formalization of mathematics. J. Symb. Logic **19** (1954), 241–266 (1955).

This paper deals with a central problem in the foundations of analysis (including sets of points), and is therefore reviewed in some detail. The author describes a theory Σ of sets (of integers) which is intended (i) to avoid impredicative notions (quantification over all sets) and thereby to ensure consistency, (ii) to avoid Gödel-incompleteness, (iii) to permit the development of all of Cantor's set theory. The schema for Σ is based on the informal notion of a constructive ordinal (c.o.) α: Σ_0 is a typical system of arithmetic; Σ_α, whose variables x_α have the order α, is a weak quantification theory of class-membership; its sets are defined by quantification over the sets in Σ_β with $\beta<\alpha$. He also allows variables to range over Σ: e.g. if $A(x, y)$ contains no bound variables of order >0, $\vdash(x)(Ey)A(x, y)$ if and only if: for each α there exist ordinals β and γ such that $\vdash_\gamma(x_\alpha)(Ey_\beta)A(x_\alpha, y_\beta)$ (where the existential quantifiers β and γ are to be interpreted constructively). In support of (iii) the author mentions that the principle of the least upper bound and some theorems on measure have analogues in Σ. It is evident that he is after a set theory whose integers have a standard model: otherwise (i) could be satisfied trivially, since every consistent set theory has an arithmetic (predicative) model in $P_2\cap Q_2$. Condition (ii) is to be achieved by leaving the notion of a c.o. informal.

The reviewer is unable to reconcile the requirements (i), (ii), (iii) with the notion of a totality of c.o. The notion of a well-ordering is itself impredicative: all descending sequences must be finite, not only those which are definable by specified methods; further, there is no hope of finding systematically, even for given (recursive) orderings R, an α

such that R is a well-ordering with respect to all sequences of Σ, if all descending sequences of Σ_a are finite. If only those c.o. are considered which can be labelled by some effective notation for ordinals, (ii) is violated. In fact, very stringent requirements on such a notation are needed if the proof predicate for Σ_a is to be decidable. If undecidable proof predicates are envisaged it is unreasonable to interpret the quantifiers β, γ in the definition of "$\vdash(x)(Ey)A(x,y)$" constructively; a more hopeful line is this: for each α a contradiction follows from the assumption that, for each β and γ, "$(x_a)(Ey_\beta)A(x_a, y_\beta)$" cannot be proved in Σ_γ.

The examples concerning (iii) are not conclusive. Measurable sets and their complements can be simultaneously covered by open sets (sets of intervals) with little overlap, and, hence, the simpler results on measurable sets are established by proving corresponding results about open sets; but though the notion of an interval is impredicative ("all" points between the end-points), the constructions use only the end points; in other words, the results concern sequences, and not arbitrary sets, of real numbers. The (translation into Σ of the) standard proof of the Cantor-Bendixson theorem (CB) by means of the transfinite series of derived sets [e.g. Littlewood, Elements of the theory of real functions, 2nd ed., Heffer, Cambridge, 1926] meets the following difficulty: if $F^{(\beta)}$ is the βth derived set of the (closed) set F, the order of $F^{(\beta)}$ is $\geqq\beta$, and therefore the transfinite series cannot be defined in Σ_a for $\beta>\alpha$. [CB is of the form "$(x)(Ey)A(x,y)$" since a closed set is determined by the end points of its complementary intervals.] A similar difficulty arises with Besicovitch's fundamental sequence in the proof of the implication: from the continuum hypothesis follows the existence of a rarified set [Acta Math. 62, 289–300 (1934)]. The possibility of using predicative definitions in the simpler parts of the theory of sets of points is analogous to the use of computable functions in convergence theory: though they occur in large parts of this theory, the characteristic feature which distinguishes it from school mathematics, is the use of non-effective definitions. A theory which ignores the latter is not regarded as a foundation for convergence theory.

The schema Σ is related to Gödel's class V of constructible sets [Consistency of the continuum hypothesis, Princeton, 1940; MR 2, 66] and Lorenzen's papers on classical analysis [e.g. Math. Z. 54, 1–24, 275–290 (1951); MR 13, 310, 615], which are the only published evidence on the author's claim (iii). The criticism of the author's use of c.o. does not apply to V since Gödel uses a theory of ordinals based on an impredicative set theory, nor to Lorenzen since he deals with particular theorems and uses very simple ordinals (ω, ω^2; or more generally $\alpha+\omega$, $\alpha+\omega^2$). In the reviewer's opinion Lorenzen's results are completely neutral on the author's problem of predicative versus impredicative set theory: from the point of view of the latter they constitute an improvement of the standard theorems of analysis; e.g., for the proof of the uniform continuity of any function which is continuous at each point of a set S, S need not be closed (contain all limit points), but, for each function f, need only contain the rationals and the limit of a specified convergent sequence of rationals; if f happens to be defined in Σ_a the convergent sequence can be defined in Σ_a too.

G. Kreisel (Reading).

Schütte, Kurt. Ein widerspruchsloses System der Analysis auf typenfreier Grundlage. Math. Z. 61, 160–179 (1954).

The author describes a type-free consistent system of analysis based on the system of logic due to Ackermann [Math. Z. 55, 364–384 (1952); 57, 155–166 (1953); MR 14, 344, 834]. A new symbol B is introduced which plays the role of a fictitious provability with the scheme $\dfrac{B(\mathfrak{F})}{\mathfrak{F}}$. Also $\bar{B}(\mathfrak{F})\vee B(\mathfrak{F})$ for any formula $\mathfrak{F}$, even though the law of the excluded middle does not generally hold in the system. A term $\mathfrak{m}$ defines a set (written $M(\mathfrak{m})$), if the law of the excluded middle holds for $\mathfrak{m}(t)$ (t a term) and the following stronger condition holds:

$$M(\mathfrak{m})\equiv(x)B[\{\bar{\mathfrak{m}}(x)\vee B(\mathfrak{m}(x))\}\wedge\{\mathfrak{m}(x)\vee B(\bar{\mathfrak{m}}(x))\}].$$

Using this notion it is possible to introduce the natural numbers, the real numbers and even to show that the various sets of the branched theory of types are included.

I. N. Gál (Ithaca, N. Y.).

Collins, George E. Distributivity and an axiom of choice. J. Symb. Logic 19 (1954), 275–277 (1955).

In dem Formalismus aus Quines "Mathematical logic" [Harvard Univ. Press, 1951; MR 13, 613] wird die Äquivalenz eines Auswahlaxioms mit der vollständigen Distributivität von Durchschnitt und Vereinigung bewiesen.

P. Lorenzen (Bonn).

Lacombe, Daniel. Classes récursivement fermées et fonctions majorantes. C. R. Acad. Sci. Paris 240, 716–718 (1955).

Im Anschluss an Kleene und Post [Ann. of Math. (2) 59, 379–407 (1954); MR 15, 772] gibt Verf. zunächst eine neue Definition der Unentscheidbarkeitsgrade. Ausgehend von allen endlichstelligen zahlentheoretischen Funktionen, heisst eine Klasse $\mathfrak{F}$ solcher Funktionen "monogen", wenn es eine Funktion φ gibt, derart dass $\mathfrak{F}$ die Klasse aller bzgl. φ rekursiven Funktionen ist. Die monogenen Klassen bilden (bzgl. $\subseteq$) einen Halbverband, der isomorph zum Halbverband der Unentscheidbarkeitsgrade ist. Eine Klasse $\mathfrak{F}$ heisst "rekursiv abgeschlossen" (r.a.), wenn es eine Klasse $\mathcal{S}$ gibt, derart dass $\mathfrak{F}$ die Klasse aller bzgl. $\mathcal{S}$ rekursiven Funktionen ist. Ist $\mathfrak{F}$ r.a., dann wird eine "Erweiterung" $\mathfrak{F}'$ definiert. Für ein 2-stelliges Prädikat p, dessen characteristische Funktion zu $\mathfrak{F}$ gehört, sei q definiert durch $q(x)\leftrightarrow(\exists y)p(x,y)$. $\mathfrak{F}'$ sei die r.a. Hülle der Klasse der charakteristischen Funktionen dieser q. $\mathfrak{F}^*$ sei die Vereinigung von $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{F}''$, $\cdots$. Verf. formuliert zwei Sätze über diese Operationen $'$ und *, von denen einer zum Beweis der Sätze aus seiner früheren Note [C. R. Acad. Sci. Paris 239, 1108–1109 (1954); MR 16, 555] gebraucht wird.

P. Lorenzen (Bonn).

Goodstein, R. L. The recursive irrationality of π. J. Symb. Logic 19 (1954), 267–274 (1955).

A (primitive, general) recursive sequence of rational numbers s_n is (p., g.) recursively convergent when there exists a (p., g.) recursive function $n(k)$ such that for $n\geqq n(k)$, $|s_n-s_{n(k)}|<2^{-k}$. The sequence is (p., g.) recursively irrational when there exist also (p., g.) recursive functions $i(p,q)$ and $N(p,q)$ such that for $n\geqq N(p,q)$, $|s_n-p/q+1|>1/i(p,q)$. The sequence is strongly (p., g.) recursively convergent in the scale r when there exists a (p., g.) recursive function $n(k)$ such that for $n\geqq n(k)$, $[r^ks_n]=[r^ks_n]$.

The author constructs a primitive recursive sequence whose limit is the zero of the Maclaurin series for sin x at π. This sequence is such that he is able to prove its primitive recursive irrationality, and to obtain a primitive recursive bound on the number of consecutive occurrences of a given block of digits in the decimal expansion of π. In the last

section of the paper the author shows the existence, for any consistent formal system $\mathfrak{R}$ adequate for recursive arithmetic, of a primitive recursive, primitive recursively convergent sequence g_n whose limit is 1, but for which there is possible in $\mathfrak{R}$ neither a proof of $\lim g_n = a$ for any rational number a, nor a proof of the primitive recursive irrationality of the sequence. ($\lim g_n = 1$ if and only if an undecidable proposition of $\mathfrak{R}$ is true.)

Reviewer's comments. 1. In a previous paper [J. London Math. Soc. **22**, 200–205 (1948); MR **9**, 404] the author showed that (p., g.) recursive irrationality implies strong (p., g.) recursive convergence in any scale. The present paper contains the observation (due to J. C. Shepherdson) that the irrationality of $\lim s_n$ implies the strong general recursive convergence of s_n in any scale. It is not hard to show in a similar way that if $\lim s_n$ is not of the form a/r^k, then s_n is strongly general recursively convergent in the scale r. Thus this property can fail only when the expansion of $\lim s_n$ in the radix r has the two (terminating and nonterminating) forms. 2. The observation of Shepherdson contains implicitly the fact that every irrational recursive real number is general recursively irrational. So the author's title might well have contained the word "primitive".

H. G. Rice (Durham, N. H.).

Dekker, J. C. E. Productive sets. Trans. Amer. Math. Soc. **78**, 129–149 (1955).

Post [Bull. Amer. Math. Soc. **50**, 284–316 (1944); MR **6**, 29] called a set α (of non-negative integers) productive if there is an effective method which provides, for each recursively enumerable (r.e.) set $\beta \subset \alpha$ an integer b such that $b \varepsilon \alpha - \beta$. The author reinterprets 'effective method' in two ways: using a sequence of sets ω_n which includes each r.e. set infinitely often, he calls α (i) productive if there exists a partially recursive function $p(n)$ such that if $\omega_n \subset \alpha$ then $p(n)$ is defined and $p(n) \varepsilon \alpha - \omega_n$; if $\omega_n \not\subset \alpha$ no restriction is imposed on the value of $p(n)$; (ii) completely productive if there exists a recursive function $f(n)$ such that, for all n, $f(n) \varepsilon (\alpha - \omega_n) \cup (\omega_n - \alpha)$. Similarly, he speaks of contraproductive sets where "ω_n" and "α" are interchanged in definition (i). Definition (ii) is not altered: completely productive = completely contraproductive. Finally, he has semiproductive sets α satisfying the condition: there exists a partially recursive function $r(n)$ such that if $\omega_n \subset \alpha$ then $\omega_n \subset \omega_{r(n)} \subset \alpha$ and $\omega_n \neq \omega_{r(n)}$. (Presumably semicontraproductive sets would be of equal interest.)

The author refers to an unpublished proof of J. R. Myhill which shows that every productive set is completely productive (the converse is evident). The paper contains an example of a contraproductive set which is not productive, and of a semiproductive set which is not contraproductive, and therefore not productive either. There are $2^{\aleph_0}$ productive sets.

Reviewer's note. The author raises the question whether there can be an effective proof of the productive character of a (productive) set α. Since definition (i) consists of an implication with an undecided premise, the answer depends on the meaning given to 'effectiveness': intuitionistically provable, recursively realizable, or recursively satisfiable; these notions are not equivalent for such implications. (i) cannot be recursively satisfiable since this requires a recursive decision method for $\omega_n \subset \alpha$, which is impossible for a productive α.

G. Kreisel (Reading).

Fraenkel, A. A. The intuitionistic revolution in mathematics and logic. Bull. Res. Council Israel **3**, 283–289 (1954).

ALGEBRA

***Mostowski, Andrzej, i Stark, Marceli. Algebra wyższa. Część trzecia. [Higher algebra. Part three.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1954. vi+273 pp. zł. 17.30.

This is the final volume of the book [for the preceding volumes see MR **15**, 594; **16**, 104]. It is intended as an introduction to Modern Algebra, is written, as are the preceding volumes, with great care and attention to didactic aspects, and contains a large number of examples and problems. Table of contents: Ch. XV, Theory of groups (introduction, simplest properties, cosets, automorphisms, quotient groups, normal sequences, direct products, structure of finite Abelian groups, groups of permutations). Ch. XVI, Rings and abstract fields (rings, fields, algebraic extensions, cyclotomic polynomials). Ch. XVII, Galois theory (Galois groups, their properties, applications to algebraic equations, determination of Galois groups), Ch. XVIII, Canonical forms of linear transformations (decomposition of linear spaces into direct sums of special invariant spaces, Jordan's canonical form, invariant factors and normal divisors, normal forms of orthogonal and unitary transformations).

A. Zygmund (Chicago, Ill.).

***Smirnow, W. I. Lehrgang der höheren Mathematik. Teil III, 1.** Deutscher Verlag der Wissenschaften, Berlin, 1954. vii+283 pp. DM 14.00.

Reviews of volumes IV (1941, 1951) and V (1947) have appeared in MR **6**, 42; **9**, 574; **14**, 145. This is a translation of the Russian fifth edition [Moscow, 1951]. A list of references and an index have been added. There are three main chapter headings: I. Determinanten und die Auflösung von Gleichungssystemen. II. Lineare Transformationen und Quadratische Formen. III. Elemente der Gruppentheorie und lineare Darstellungen von Gruppen. The material on linear algebra is fairly standard, but there is considerable detail concerning the applications to analysis, as well as an introduction to the infinite-dimensional case. An unusual item in Chapter III is a proof of the simplicity of the orthogonal group and of the Lorentz group. The book concludes with a 28-page introduction to Lie groups, the emphasis being on examples.

I. Kaplansky.

Devidé, Vladimir. Ein Problem über Wägen. Elem. Math. **10**, 11–15 (1955).

Given $(3^n - 1)/2$ coins, of which at most one is bad, and a single additional good coin, it is possible to determine in n weighings on a beam balance which coin, if any, is bad, and whether it is heavy or light. The author was not aware that the result is known [Fine, Amer. Math. Monthly **54**, 489–490 (1947)].

N. J. Fine (Philadelphia, Pa.).

Rashevsky, N. Note on a combinatorial problem in topological biology. Bull. Math. Biophys. **17**, 45–50 (1955).

In connection with a previous paper, an expression is derived for the number of possibilities in which n distinguishable elements can be distributed into $m \leq n$ classes, so that each class contains at least one element.

Author's summary.

Golomb, S. W. Checker boards and polyominoes. Amer. Math. Monthly 61, 675–682 (1954).

Concerned with dividing the squares of an 8×8 checker board into various configurations formed by a smaller number of squares, which are called polyominoes, and of which dominoes (two squares side by side) are a special case.

G. A. Dirac (Vienna).

Meier, Paul. Analysis of simple lattice designs with unequal sets of replications. J. Amer. Statist. Assoc. 49, 786–813 (1954).

If k^2 varieties are arranged in a square array, one pattern of a simple lattice design consists of taking the rows as blocks and the other pattern of taking the columns as blocks. In the usual simple lattice, these patterns are replicated an equal number of times, but in the present paper, they are replicated unequally. The raison d'être of the unequal sets of replicates rests on comparison with the more balanced triple, quintuple, etc., lattices. The unequal sets, though slightly less efficient, require considerably less computational labor.

The analysis of variance is developed in such a way that exact tests of significance are available for both block and varietal effects. Intra- and inter-block estimates of varietal means, and variances of varietal differences, are presented. The efficiency of the unequal sets designs is calculated relative to randomized blocks, lattices with equal sets, and triple and quintuple lattices. A synthetic example, fully worked out, exemplifies the computations.

W. S. Connor (New Brunswick, N. J.).

Youden, W. J., and Connor, W. S. The chain block design. Biometrics 9, 127–140 (1953).

A chain block design is an arrangement of r treatments into b blocks in the following manner. A treatment is said to belong to the class c_1 if it is replicated once or to the class c_2 if it is replicated twice. The treatments of c_2 are divided into b groups of $n_2 \geqq 1$ treatments each. The treatments of the jth group occur in the jth and $(j+1)$th (mod b) block and nowhere else. The authors give the analysis of variance of this design and illustrate it by an example. The chain block designs are appropriate for measurement with a small standard error.

H. B. Mann (Columbus, Ohio).

Mandel, John. Chain block designs with two-way elimination of heterogeneity. Biometrics 10, 251–272 (1954).

A new design is presented which generalizes an idea of Youden and Connor in the paper reviewed above. The analysis of variance of this new design is derived and illustrated by an example.

H. B. Mann.

Roy, Purnendu Mohon. On the method of inversion in the construction of partially balanced incomplete block designs from the corresponding b.i.b. designs. Sankhyā 14, 39–52 (1954).

As introduced by F. Yates [Ann. Eugenics 7, 121–140 (1936)], a balanced incomplete block (b.i.b.) design is an arrangement of v varieties in b blocks of size k $(k<v)$, in such a way that no variety occurs more than once per block, every variety occurs in r blocks, and every pair of varieties occurs in λ blocks. A partially balanced incomplete block (p.b.i.b.) design is a generalization of a b.i.b. design to admit several λ's [Bose and Nair, Sankhyā 4, 337–372 (1939)]. The method of inversion consists of obtaining a new design by regarding varieties as blocks and blocks as varieties.

It is proved in this paper that the following b.i.b. designs, when inverted, are p.b.i.b. designs: (1) b.i.b. designs with $\lambda=1$ $(v\neq b)$; (2) b.i.b. designs derived from the b.i.b. designs $v=b$, $r=k$, $\lambda=2$ by deletion from them of all varieties occurring in some block; (3) b.i.b. designs for which the blocks are separable into r sets of n blocks each, such that each set contains every variety once; (4) b.i.b. designs in which the blocks consist of all combinations of the v varieties taken k at a time.

The complement of a given b.i.b. design B is the b.i.b. design which is obtained by replacing the varieties in each block of B by the remaining varieties. It is proved that the inverse of B is a p.b.i.b. design whose complement is the inverse of the complement of B.

W. S. Connor.

Nair, K. R. The so-called almost-balanced incomplete block designs. Calcutta Statist. Assoc. Bull. 5, 181–184 (1954).

The author discusses a design described by M. H. Quenouille [Statistics and mathematics in biology, Iowa State College Press, 1954, Chap. 11, pp. 149–158; MR 16, 56] and named almost-balanced incomplete block design. The author shows that this design is a partially balanced incomplete block design with three associate classes and an efficiency factor of only 0.408. The author enumerates 7 other designs of a similar nature all of them with higher efficiency factors. It may be observed that all these designs have efficiency factors less than 0.55.

H. B. Mann.

Shrikhande, S. S. The non-existence of certain affine resolvable balanced incomplete block designs. Canad. J. Math. 5, 413–420 (1953).

The parameters of an affine resolvable balanced incomplete block design (ARBIB) can be expressed by two parameters n and t as follows: $v=nk=n^2[(n-1)t+1]$, $b=nr=n(n^2t+n+1)$, $\lambda=nt+1$. Using the theory of Hasse invariants as well as methods developed by Connor, [Ann. Math. Statist. 23, 57–71 (1952); 24, 135 (1953); MR 13, 617] the author proves the following three theorems. An ARBIB does not exist: (1) when n and t are odd and either $n[(n-1)t+1]$ is not a perfect square or $n[(n-1)t+1]$ is a perfect square, $nt\equiv1$ (mod 4) and the square-free part of n contains a prime $\equiv3$ (mod 4); (2) when n is odd and t even and either $(n-1)t+1$ is not a perfect or $(n-1)t+1$ is a perfect square and $n+t\equiv1$ (mod 4) and the square-free part of n contains a prime $\equiv3$ (mod 4); (3) when $n\equiv2$ (mod 4) and the square-free part of n contains a prime $\equiv3$ (mod 4).

The existence of a complete orthogonal array

$$(\mu n^2, (\mu n^2-1)/(n-1), n, 2)$$

implies the existence of an ARBIB with parameters

$$v=nk=\mu n^2,\quad b=nr=n\left(\frac{\mu n^2-1}{n-1}\right),\quad \lambda=\frac{\mu n-1}{n-1}$$

and conversely. Hence the authors results also show the non-existence of certain orthogonal arrays.

H. B. Mann (Columbus, Ohio).

Fan, Ky, and Todd, John. A determinantal inequality. J. London Math. Soc. 30, 58–64 (1955).

Let $\begin{pmatrix} a_1, \cdots, a_n \\ b_1, \cdots, b_n \end{pmatrix}$ be a matrix no two-column minor of which is singular. Let (p_{ij}) be a symmetric matrix, the sum $2P$ of the non-diagonal elements of which is not 0. Then

the inequality

$$P^2\left[\sum a_i^2\right]\left[\sum a_i^2\sum b_i^2-\left(\sum a_ib_i\right)^2\right]^{-1} \leqq \sum_i\left[\sum_{j\neq i} p_{ij}a_j(a_jb_i-a_ib_j)^{-1}\right]^2$$

holds. The case $p_{ij}\equiv 1$, $a_i=\cos\theta_i$, $b_i=\sin\theta_i$ was discovered by J. B. Chassan [Amer. Math. Monthly **62**, 353–356 (1955)]. The generalization of the above from an $n\times 2$ to an $n\times m$ array is given also. The brackets on the left sides are Gramians; each quotient on the right side is the quotient of an $(m-1)^2$ minor determinant by an m^2 minor determinant. The following minimum problem is solved. Let a^i be linearly independent vectors in unitary n-space $(i=1, 2, \cdots, m)$. Let the vector x vary so that the relations $(a_1, x)=(a_2, x)=\cdots=(a_{m-1}, x)=1-(a_m, x)=0$. Then the minimum of (x, x) is

$$\text{Gramian }(a^1, \cdots, a^{m-1})/\text{Gramian }(a^1, \cdots, a^m).$$

J. L. Brenner (Pullman, Wash.).

Littlewood, D. E. Skew-symmetric determinants. Math. Gaz. **39**, 57–58 (1955).

Koschmieder, Lothar. Sui determinanti ortosimmetrici di funzioni trigonometriche e iperboliche. Boll. Un. Mat. Ital. (3) **9**, 266–270 (1954).

Inspired by recent work of Turán [Časopis Pešt. Mat. Fys. **75**, 113–122 (1950); MR **12**, 824] and others, the author proves that

$$\begin{vmatrix}\sin[\alpha+(k+q)\beta]x & \sin[\alpha+(k+r)\beta]x\\ \sin[\alpha+(l+q)\beta]x & \sin[\alpha+(l+r)\beta]x\end{vmatrix} = -\sin(k-l)\beta x\sin(q-r)\beta x,$$

and thus establishes an inequality of Turán type:

$$\begin{vmatrix}\sin\alpha x & \sin(\alpha+\beta)x\\ \sin(\alpha+\beta)x & \sin(\alpha+2\beta)x\end{vmatrix} = -\sin^2\beta x\leqq 0.$$

Also, if we let $a_{ij}=\sin[\alpha+(i+j-2)\beta]x$ $(i, j=1, \cdots, m)$, then it is proved that $\det(a_{ij})=0$ for $m\geqq 3$. Both results hold also for cosines. *G. E. Forsythe.*

Have

***Woodbury, Max A. Properties of Leontief-type input-output matrices.** Economic activity analysis, pp. 341–363. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

This paper gives an account of theorems, both classical and modern, that are applicable to the type of input-output matrices used by Leontief. Among the topics dealt with are: determinantal bounds, characterization of singular Leontief-type matrices, and the positive nature of the inverses of Leontief matrices. *H. W. Kuhn* (Bryn Mawr, Pa.).

***Woodbury, Max A. Characteristic roots of input-output matrices.** Economic activity analysis, pp. 365–382. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

This paper may be considered as an extension of a previous article by the writer (see review above). In the present paper four related topics are considered: (1) properties of characteristic roots of positive and non-negative matrices, (2) characteristic roots of input-output and related matrices, (3) the relation of the characteristic roots and the expansion rate in a simple linear economy model, (4) application of the results to the calculation of the roots of an input-output matrix. (Author's summary.)

H. W. Kuhn (Bryn Mawr, Pa.).

***Bott, R., and Mayberry, J. P. Matrices and trees.** Economic activity analysis, pp. 391–400. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

Let $A=(a_{ij})$ be an $n\times n$ matrix with complex entries and let Γ be the complete oriented graph with vertices $0, 1, \cdots, n$. Define f_A, a complex-valued function on the edges of Γ, by $f_A(i, j)=-a_{ij}$ for $i\neq j$, $f_A(i, 0)=0$, and $f_A(0, j)=\sum_i a_{ij}$, for $i, j=1, \cdots, n$. A subgraph T of Γ that is a tree is said to be rooted if 0 is connected to every vertex of Γ by an oriented path in T. If T is a rooted tree, then $v_A(T)$ is the product of all values of f_A on the edges of T. Matrix-Tree Theorem: The determinant of A equals $\sum v_A(T)$, where the sum is taken over all rooted trees in Γ. (An error in the proof of Lemma 4 has been noted and corrected by R. H. Crowell in "Invariants of alternating link types", Thesis, Princeton, 1955.) *H. W. Kuhn* (Bryn Mawr, Pa.).

***Whitin, T. M. An economic application of "Matrices and trees".** Economic activity analysis, pp. 401–408. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

In this paper, the Matrix-Tree Theorem (see the preceding review) is applied to show that, under reasonable economic assumptions, Leontief input-output matrices either have a positive determinant or there are closed sub-economies present. It should be noted that Figures 6 and 7 were interchanged in printing this paper. *H. W. Kuhn.*

Maxfield, John E., and Gardner, Robert S. Note on linear hypotheses with prescribed matrix of normal equations. Ann. Math. Statist. **26**, 149–150 (1955).

The authors prove that to each symmetric positive semi-definite matrix A of integers there exist an integer a and a matrix B of integers such that $BB^T=a^2A$, where B^T is the transpose of B. *D. M. Sandelius* (Göteborg).

Stein, P. An extension of a formula of Cayley. Proc. Edinburgh Math. Soc. (2) **9**, 91–99 (1954).

Let x_{ik} $(i, k=1, 2, \cdots, n)$ be n^2 independent variables, a_{ik} a set of constants and ξ_{ik} the operators $a_{ik}\partial/\partial x_{ik}$. The determinant of the x_{ik} is denoted by x, and x_{IK} stands for the minor of x specified by the row suffixes $I=(i_1, \cdots, i_m)$ and the column suffixes $K=(k_1, \cdots, k_m)$, where $m\leqq n$. The cofactor of x_{IK} in x is written x^{IK}. An analogous notation is used for the determinant of the ξ_{ik} and its minors.

The problem is to evaluate $\xi^{IK}x^{\alpha}$, where α is a positive integer. The author derives an explicit, though somewhat complicated, formula for this expression. From it he deduces a classical result of Capelli which refers to the case $a_{ik}=1$, and which in turn generalizes a theorem of Cayley. Similar formulae can be obtained when the determinants ξ or x or both are replaced by the corresponding permanents.

W. Ledermann (Manchester).

Cherubino, Salvatore. Permutabilità e logaritmi delle matrici. Rend. Mat. e Appl. (5) **14**, 221–238 (1954).

It is shown that any square matrix A with complex elements can be expressed uniquely in the form A_D+A_J such that if $H^{-1}AH$ is the Jordan canonical form of A then

$H^{-1}A_DH$ is the diagonal matrix whose leading diagonal coincides with that of $H^{-1}AH$, while $H^{-1}A_JH$ is triangular and nilpotent. This decomposition does not depend upon the choice of H nor upon the particular canonical form referred to. Further, this is the only decomposition of A as the sum of two commuting matrices such that one term has simple elementary divisors and has the same eigenvalues as A, while the other is nilpotent.

If A and B commute, then A_D, A_J, B_D, B_J all commute. Also, if $\phi(z_1, \cdots, z_p)$ is a rational function of several complex scalars and if $X^{(1)}, \cdots, X^{(p)}$ are a set of commuting matrices each of order n, then

$$[\phi(X^{(1)}, \cdots, X^{(p)}]_D = \phi(X_D^{(1)}, \cdots, X_D^{(p)}).$$

If $|A| \neq 0$ and A has eigenvalues $\alpha_1, \cdots, \alpha_n$, the principal value of $\lg A$ is given by $\lg A = \lg A_D - \sum_{s=1}^{n-1} (-A_D^{-1}A_J)^s s^{-1}$, where $\lg A_D = H \cdot \operatorname{diag}[\lg \alpha_1, \cdots, \lg \alpha_n] \cdot H^{-1}$. With this notation, $e^{\lg A} = A$ and $(\lg A)_D = \lg A_D$. Further, if A, B are elements of an algebra of commuting non-singular complex matrices, then $\lg (AB) = \lg A + \lg B$.

D. E. Rutherford (St. Andrews).

Benedicty, Mario. Interi caratteristici e divisori elementari delle matrici normali di Severi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 716–720 (1954).

Sia ω una matrice quasi abeliana di caratteri p, δ_1, δ_2, ρ, ossia una matrice equivalente ad un'altra avente la forma normale di Severi [Pont. Accad. Sci. Scripta Varia 4 (1947); MR 9, 578], intendendo che due matrici simili ω, ω^* siano equivalenti se esistono una matrice complessa non degenere β ed una matrice intera unimodulare B tali che $\omega^* = \beta\omega B$. In un precedente lavoro [Rend. Mat. e Appl. (5) 12, 332–339 (1954); MR 15, 821] l'Autore ha posto il problema di determinare le forme normali cui è equivalente una prefissata matrice quasi abeliana, e cioè di determinare gli interi caratteristici p, δ_1, δ_2, φ ed i divisori elementari $d_1, d_2, \cdots, d_p$ di tutte le matrici quasi abeliane equivalenti ad una prefissata matrice normale di Severi. L'Autore risolve qui completamente questo problema quando la caratteristica della matrice considerata sia uguale al numero delle colonne o ne differisca di una unità; negli altri casi il problema è risolto sotto certe ipotesi di genericità. Dopo aver osservato che se p^*, δ_1^*, δ_2^*, ρ^*; p, δ_1, δ_2, ρ sono gli interi caratteristici di due matrici normali di Severi tra loro equivalenti risulta: $p^* = p - k$, $\delta_1^* = \delta_1 + 2k$, $\delta_2^* = \delta_2 - k$, $\rho^* = \rho - k$ ove k è un intero tale che $-\delta_2/2 \leqq k \leqq \rho$ (sicchè gli interi caratteristici di una matrice quasi abeliana dipendono al più dall'indeterminazione di k), l'Autore dimostra che: a) se $p = \rho$ ogni matrice quasi abeliana è equivalente ad una matrice normale di Severi per cui: $p^* = 0$, $\delta_1^* = \delta_1 + 2\rho$, $\delta_2^* = \delta_2 - \rho$, $\rho^* = 0$; b) se $p = \rho - 1$, è equivalente ad una matrice normale di Severi con: $p^* = 1$, $\delta_1^* = \delta_1 + 2\rho$, $\delta_2^* = \delta_2 - \rho$, $\rho^* = 0$; c) se $\rho > p + 1$ una matrice quasi abeliana generica è equivalente soltanto a matrici normali di Severi aventi i medesimi interi associati ed i medesimi divisori elementari.

D. Gallarati (Genova).

Sinden, Frank W. An oscillation theorem for algebraic eigenvalue problems and its applications. Mitt. Inst. Angew. Math. Zürich no. 4, 57 pp. (1954).

Let X, Y denote vectors with n real components. The variation number, $V(X)$, is defined to be the number of sign changes in the sequence $(x_1, \cdots, x_n)$ of the components of X. The re l matrix A is said to be variation-diminishing if $V(Y) \leqq V(X)$ (variation-increasing if $V(Y) \geqq V(X)$) for every vector X, when Y is given by $AX = Y$.

The eigenvalue problem (*) $AX = \lambda DX$, where D is a diagonal matrix with positive elements is considered under the hypothesis (a) A is symmetric and positive definite, (b) A is variation-diminishing, and (c) the codiagonal elements $a_{i-1,i}$, $a_{i+1,i}$ of A are non-zero.

The main theorem states that, under conditions (a), (b), and (c): 1) the eigenvalues λ_i of (*) are simple and positive and may be ordered: $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$; 2) if X_i denotes the eigenvector corresponding to λ_i, then $V(X_i) = i - 1$; 3) if X is an eigenvector with components $(x_1, \cdots, x_n)$, then $x_1 \neq 0$, $x_i = 0$ only if $x_{i-1}x_{i+1} < 0$, $x_n \neq 0$. As a corollary, the theorem is shown to hold under the conditions (a) and (c) with the matrix A variation-increasing except that the variation number of X_i as given in 2) above is replaced by $n - i$.

Application is made to the general class of matrices arising from the finite-difference approximations of formally self-adjoint ordinary differential equations under mild restrictions on the coefficients and boundary conditions. The specific example of the vibrating rod with intermediate supports is carried out in detail.

A further refinement of the theory permits an easy determination of lower and upper bounds for each of the eigenvalues.

E. Isaacson (New York, N. Y.).

Šostak, R. Ya. On a criterion of conditional definiteness of a quadratic form of n variables, subject to linear relations, and on a sufficient condition for a conditional extremum of a function of n variables. Uspehi Mat. Nauk (N.S.) 9 no. 2(60), 199–206 (1954). (Russian)

The author proves that the quadratic form $\sum a_{ij}x_ix_j$ is positive definite when the side conditions $\sum b_{li}x_i = 0$ are satisfied if and only if all the principal diagonal minors, beginning with the minors of order $2k+1$, in the determinant

$$D = (-1)^k \begin{vmatrix} 0 & B \\ B^t & A \end{vmatrix}$$

are positive. Here B^t denotes the transpose of the $k \times n$ matrix B, A is $n \times n$, and so D is $(k+n) \times (k+n)$. This result he applies to the study of the second variation in the standard Lagrange multiplier problem.

J. M. Danskin.

Dolph, C. L., McLaughlin, J. E., and Marx, I. Symmetric linear transformations and complex quadratic forms. Comm. Pure Appl. Math. 7, 621–632 (1954).

Consider an n by n matrix $A = (a_{jk})$ of complex elements which is symmetric $a_{jk} = a_{kj}$, but not Hermitian symmetric, and which has simple elementary divisors. The authors deal with the problem of characterizing the eigenvalues $\lambda_1, \cdots, \lambda_n$, of A in terms of the quadratic form $Q(z) = \sum\sum a_{jk}z^jz^k$. First, the $\lambda_1, \cdots, \lambda_n$ are shown to be the stationary values of $Q(z)$ under the condition $\sum z^jz^j = 1$; second, under the conditions $\mu_j = \operatorname{Im} \lambda_j > 0$, the numbers $\mu_1, \cdots, \mu_n$ are characterized by certain mini-maximum conditions on the real, $2n$-dimensional form $\operatorname{Im} Q(z)$.

P. Hartman.

Reiner, Irving. Symplectic modular complements. Trans. Amer. Math. Soc. 77, 498–505 (1954).

The problem solved in the paper is the following: Find a necessary and sufficient condition under which a matrix with rational integral elements can be completed to be a symplectic modular matrix, and if the condition is satisfied, find all possible complements.

L. K. Hua (Peking).

Jacob, Henry G., Jr. Coherence invariant mappings on Kronecker products. Amer. J. Math. **77**, 177–189 (1955).

This paper contains a generalization of a theorem on matrices by the reviewer [see J. Chinese Math. Soc. **1**, 109–163 (1951)]. The reviewer has characterized the class of 1-1 mappings of the totality of $n\times m$ $(n, m\geqq 2)$ matrices, over a division ring which preserves coherence, while the present author treats the case of the Kronecker product of two spaces of arbitrary dimension over a division ring. The methods used in the present paper are analogous to those of the reviewer.

The paper also contains an application to the determination of the isomorphisms between primitive rings with minimal ideals. *L. K. Hua* (Peking).

Witt, Ernst. Über eine Invariante quadratischer Formen mod 2. J. Reine Angew. Math. **193**, 119–120 (1954).

The invariant Δ mod $(\gamma^2+\gamma)$ [same J. **183**, 148–167 (1941); MR **4**, 237] of a quadratic form $\sum_{i=1}^{2m} a_{ii}x_i^2+\sum_{i<k} a_{ik}x_ix_k$ in a field K of characteristic 2 is expressed as a rational function of the coefficients and a new and very simple proof of its invariance is given. For this purpose the author considers the field K as a residue field mod 2 in a domain of integrity of characteristic 0. He puts $A=(\alpha_{ik})$ with $\alpha_{ii}=2a_i$, $\alpha_{ik}=\alpha_{ki}=a_{ik}$ and $A^*=(\alpha^*_{ik})$ with $\alpha^*_{ii}=0$, $\alpha^*_{ik}=-\alpha^*_{ki}=a_{ik}$. Then Δ is given by $(-1)^m|A|\,|A^*|^{-1}=1+4\Delta$. *C. Arf.*

Klingenberg, Wilhelm, und Witt, Ernst. Über die Arfsche Invariante quadratischer Formen mod 2. J. Reine Angew. Math. **193**, 121–122 (1954).

The authors consider the quadratic form $\mathbf{x}'A\mathbf{x}$ in a field of characteristic 2. Assuming that $|A+A'|\neq 0$, a choice of the variables can be made in such a way that the matrices $A+A'$ and A assume the forms

$$\begin{pmatrix}0 & E\\ E & 0\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}P & E\\ 0 & Q\end{pmatrix}.$$

Using the usual rules for the computing of traces, it is shown that Tr (PQ) mod $(\gamma^2+\gamma)$ is invariant with respect to the substitutions of the symplectic group. This invariant Tr (PQ) mod $(\gamma^2+\gamma)$ is the same as the one given by Arf [same J. **183**, 148–167 (1941); MR **4**, 237]. *C. Arf.*

Kneser, Martin. Bestimmung des Zentrums der Cliffordschen Algebren einer quadratischen Form über einem Körper der Charakteristik 2. J. Reine Angew. Math. **193**, 123–125 (1954).

The Clifford Algebra $\mathfrak{A}$ of a quadratic form $q(\mathbf{x})$ in a field K of characteristic 2 is defined independently of the choice of the variables. The products uv of the usual generators of $\mathfrak{A}$ generate a subalgebra the center of which is shown to be the residue ring $K[x]/(x^2+x+\Delta)$, where Δ mod $(\gamma^2+\gamma)$ is the invariant of the form $q(\mathbf{x})$ given by Arf [same J. **183**, 148–167 (1941); MR **4**, 237]. Thus an elegant proof of the invariance of Δ mod $(\gamma^2+\gamma)$ is given. *C. Arf* (Istanbul).

Csakalov, L. On two factor series appearing in the theory of algebraic equations. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 343–352 (1954). (Hungarian)

For given real $c_0, \cdots, c_n$ we seek α, β such that every polynomial $\sum a_ix^i$ with $\sum a_ic_i=0$ which is $\geqq 0$ for $\alpha\leqq x\leqq\beta$ vanishes identically. Such α and β exist if and only if the form $\sum c_{p+q}u_pu_q$ $(p, q\leqq k-1=[\frac{1}{2}n])$ is positive definite (without saying so the author gives thus a characterization of the convex hull of the norm curve

$$(x_1, x_2, \cdots, x_n)=(t, t^2, \cdots, t^n));$$

max α and min β are, for odd $n>1$, the smallest and largest root of

$$\begin{vmatrix} 1 & x\cdots x^k\\ c_0 & c_1\cdots c_k\\ c_1 & c_2\cdots c_{k+1}\\ \cdots & \cdots\\ c_{k-1} & c_k\cdots c_{2k-1}\end{vmatrix}=0.$$

[Cf. L. Tchakaloff, Acta Math. **63**, 77–97 (1934); and A. K. Haradze, Mean value theorems applied to polynomials (Russian), Tbilisi, 1947.] *T. S. Motzkin.*

Abstract Algebra

Have

★Hermes, Hans. Einführung in die Verbandstheorie. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXIII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+164 pp. DM 22.80.

A most attractive introduction to the ideas and results of lattice theory is achieved by a careful selection of theorems having elegant proof. The book is comparable to "Leçons sur la théorie des treillis . . ." [Gauthier-Villars, Paris, 1953; MR **15**, 279], by M. L. Dubreil-Jacotin, L. Lesieur, and R. Croisot, but even more expository in character. Even though it contains no new results, it is so well-written that a summary of its contents seems desirable.

The concepts of partly ordered set, lattice, homomorphism, and complete lattice are first introduced, and illustrated by typical examples. Then modular, distributive, and complemented lattices are distinguished, and embedded in complete lattices by ideals and by cuts. A long chapter on modular lattices follows; it includes modern discussions of projective geometries, partition lattices, permutable partitions, the Schreier-Zassenhaus Theorem, and abstract linear dependence. Next, distributive lattices and Boolean algebras are related to rings of sets and Boolean spaces; infinite distributivity is also treated. The book closes with discussions of Zorn's Lemma, congruence relations in lattices, two-valued logics, and universal algebra. *G. Birkhoff.*

★Dubreil, Paul. Les relations d'équivalence et leurs principales applications. Les conférences du Palais de la Découverte, série A, no. 194. Université de Paris, Paris, 1954. 22 pp. 88 francs.

This article contains a brief survey of the theory of equivalence relations, first in an arbitrary set and then in algebraic systems. Here the concept of regularity of an equivalence relation relative to an algebraic operation leads to the general homomorphism theorem. (A relation R is regular relative to left multiplication if aRb implies $xaRxb$ for all x.) The applications discussed include successive construction of the rational, real and complex numbers starting from the integers, the theory of distinguished subalgebras of the common algebraic systems, and some further applications to semi-groups and groups. A bibliography is given. *D. C. Murdoch* (Vancouver, B. C.).

Dwinger, Ph. On the closure operators of a complete lattice. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 560–563 (1954).

The lattice of all closure operators on a complete lattice L is distributive, if and only if L is a chain; it is a Boolean algebra if and only if L is an ordinal with unit element.

G. Birkhoff (Cambridge, Mass.).

Dwinger, Ph. On the closure operators of the ordinal product of closed lattices. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17**, 36–40 (1955).

Let L be a closed lattice with the set C_L of closure operators, and let L be the ordinal product (i. e., ordered pairs, lexicographically ordered) of closed lattices M and N. Then C_L and C_M are completely (i. e., even for infinitely many factors) meet-homomorphic, and some closed sublattice C_L' of C_L is completely lattice-homomorphic with C_M. In fact, the meet-homomorphism is $\phi \rightarrow \phi'$, where ϕ' is determined by $\phi(Y, O_2) = (\phi'(Y), Z)$ for $Y \varepsilon M$, Z a fixed element of N; and C_L' is the set of ϕ with $\phi(Y, Z) = (\phi'(Y), Z')$ for all $Y \varepsilon M$, $Z \varepsilon N$, Z' a fixed element of N.

P. M. Whitman (Silver Spring, Md.).

Yamamoto, Koichi. Logarithmic order of free distributive lattice. J. Math. Soc. Japan **6**, 343–353 (1954).

Let $f(n)$ denote the number of elements in the free distributive lattice with n generators. The author refines an earlier guess of Morgan Ward to the conjecture that $\log_2 f(n) \sim (2/\pi n)^{\frac{1}{2}} 2^n$. He proves the weaker result that, for every $\delta > 0$, and sufficiently large n,

$$2^n n^{-1-\delta} < \log_2 f(n) < 2^n n^{-1+\delta}.$$

G. Birkhoff (Cambridge, Mass.).

Jordan, Pascual, und Böge, Werner. Zur Theorie der Schrägverbände. II. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1954**, 79–92 (1954).

Further theorems and examples on skew-lattices [for part I, see Jordan and Witt, same Abh. **1953**, 223–232 (1953); MR **15**, 595]. The best generalization of the modular law is found to be

$$[(b \cap c) \cup a] \cap (b \cup c) = (b \cap c) \cup [a \cap (b \cup c)].$$

The concatenation of two skew-lattices $A = \{a_i\}$ and $B = \{b_j\}$ is defined as the skew-lattice of all elements of both, with the laws of A and of B plus $a_i \cap b_j = b_j \cap a_i = b_j$ and $a_i \cup b_j = b_j \cup a_i = a_i$. Cases where this operation does and does not lead to a skew-lattice of the same family are found. Certain sets are found to be sub-skew-lattices, and congruence classes are studied briefly. *P. M. Whitman.*

Benado, Mihail. Sur une généralisation de la notion de structure. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 41–48 (1953). (Romanian. Russian and French summaries)

A multilattice is a partially ordered set S in which if $a, b \varepsilon S$ have a common upper bound Ω then there is a minimal common upper bound $M \leqq \Omega$, and dually; it is not required that any common bound exist, nor that M be unique. An equivalent definition is given in terms of multi-valued operations $\wedge$ and $\vee$ and several laws thereon. Several examples are given. Some properties are studied, such as generalization of modularity (in terms of perspectivity) and Schreier refinements. *P. M. Whitman.*

Benado, Mihail. Sur la théorie de la divisibilité. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 263–270 (1954). (Romanian. Russian and French summaries)

[Cf. the preceding review.] Let G be a partially ordered group which is a multilattice; denote by H the multilattice of all $g \varepsilon G$ with $g \leqq 1$. If H is a distributive multilattice satisfying the ascending chain condition, then G is a lattice-ordered group; distributivity is defined in terms of non-existence of a five-element submultilattice of the type characteristic of non-distributive lattices. This theorem generalizes the properties of divisibility of algebraic integers in an algebraic number field. *P. M. Whitman.*

Benado, Mihail. Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. I. Čehoslovack. Mat. Ž. **4(79)**, 105–129 (1954). (French. Russian summary)

The author continues his axiomatic study of the Schreier-Zassenhaus (S-Z) refinement theorem in a partly ordered set P. He introduces a new "condition H" on quadrilaterals in P, and a concept of "modular space"; these permit him to express various conditions for the S-Z theorem to hold. Applications are given to "division" semigroups and integral domains. *G. Birkhoff* (Cambridge, Mass.).

Almeida Costa, A. Three lectures on the general theory of rings. (Applications and complements, I.) An. Fac. Ci. Porto **37**, no. 3, 129–170 (1954) = Centro Estudos Mat. Fac. Ci. Porto. Publ. no. **34**, 42 pp. (1954). (Portuguese)

The author continues his exposition of recent developments in the theory of rings [for the last installments see same An. **36**, 169–200, 221–247 (1952); MR **15**, 773]. High spots: subdirectly irreducible modules and rings, Goldman's characterization of semisimple rings and their subrings, Jacobson's theorem on the commutativity of rings satisfying $a^{n(a)} = a$. *I. Kaplansky* (Los Angeles, Calif.).

Levitzki, Jakob. On minimal central identities. Riveon Lematematika **8**, 41–58 (1954). (Hebrew. English summary)

Let F_n be the ring of square matrices of order n over the commutative field F. A non-commutative polynomial $f(X_1, \cdots, X_m)$ with coefficients in F is said to be a central polynomial of F_n if the substitution of any set of elements of F_n for $X_1, \cdots, X_m$ in $f(X_1, \cdots, X_m)$ yields an element of the centre of F_n. For given F_n, a central polynomial of lowest possible degree is called a minimal central polynomial. The author proves that the degree of a minimal central polynomial is $2n$. For $n > 2$, the set of minimal central polynomials is determined explicitly. In this case, the minimal central polynomials can be expressed as linear combinations of certain "determinants" (A "determinant" $[X_1, \cdots, X_m]$ is the sum of the products $\pm X_{i_1} \cdots X_{i_m}$, where the sign depends on the parity of the permutation $(i_1, \cdots, i_m)$). The situation is more complex if $n = 2$. In this case, the author determines the minimal central polynomials for $m \leqq 4$. The results now depend to some extent on the characteristic p of F, the case $p = 2$ requiring special consideration. [Cf. A. S. Amitsur and J. Levitzki, Proc. Amer. Math. Soc. **1**, 449–463 (1950); MR **12**, 155.]

A. Robinson (Toronto, Ont.).

Podderyugin, V. D. A condition of orderability for an arbitrary ring. Uspehi Mat. Nauk (N.S.) 9, no. 4(62), 211–216 (1954). (Russian)

The rings in question are not necessarily associative and are allowed to have divisors of 0. The author gives a necessary and sufficient condition for there to exist a simple ordering (any non-zero element either positive or negative, the sum of two positives positive, the product of two positives either positive or 0). The condition reads: for any finite set $a_1, \cdots, a_n$ it should be possible to pick $b_i = \pm a_i$ in such a way that a sum of products of the b's vanishes only if its constituents vanish. Assuming further that there are no divisors of 0, one passes easily to the condition given by Johnson [Proc. Amer. Math. Soc. 3, 414–416 (1952); MR 13, 815]. (Reviewer's remark. The cumbersome transfinite induction used in the proof can be replaced by an application of Tychonoff's theorem to a suitable Cartesian product of two-point spaces.) *I. Kaplansky.*

Morrison, D. R. Bi-regular rings and the ideal lattice isomorphisms. Proc. Amer. Math. Soc. 6, 46–49 (1955).

A p.i.u. ring is one in which every principal ideal has a unit. This is shown to coincide with bi-regularity [R. Arens and I. Kaplansky, Trans. Amer. Math. Soc. 63, 457–481 (1949); MR 10, 7]. Moreover, for any ring, the lattice of p.i.u. ideals is isomorphic to the lattice of ideals of the ring of central idempotent elements. *R. Arens.*

Rees, D. Valuations associated with a local ring. I. Proc. London Math. Soc. (3) 5, 107–128 (1955).

Given two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a commutative ring A, $l_{\mathfrak{a}}(\mathfrak{b})$ denotes the limit of $m(n)/n$, where $m(n)$ is the greatest integer such that $\mathfrak{a}^{m(n)} \supset \mathfrak{b}^n$ [cf. Samuel, Ann. of Math. (2) 56, 11–21 (1952); MR 14, 128]. The existence of such a limit is extended to the case where the filtration $(\mathfrak{a}^n)$ of A is replaced by a general filtration, and the link with the theory of valuations and pseudo valuations is given. More particularly, if $\mathfrak{q}$ is primary for the maximal ideal $\mathfrak{m}$ of an equicharacteristic local ring A, then there exist discrete rank-one valuations $v_1, \cdots, v_k$ of the quotient ring of A such that $l_{\mathfrak{q}}(Ax) = \min_i (v_i(x))$. For two m-primary ideals $\mathfrak{q}, \mathfrak{q}'$ to satisfy $l_{\mathfrak{q}}(\mathfrak{q}') = l_{\mathfrak{q}'}(\mathfrak{q}) = 1$, it is necessary and sufficient that they have the same integral closure [a result found independently by Muhly, ibid 60, 576–577 (1954); MR 16, 213]. *P. Samuel* (Cambridge, Mass.).

Andrade Guimarães, António. Systems of linear equations over a vector space. Primordial solutions. An. Fac. Ci. Porto 37, no. 3, 171–187 (1954) = Centro Estudos Mat. Fac. Ci. Porto. Publ. no. 35, 17 pp. (1954). (Portuguese)

Expository article, which is essentially a translation of the corresponding paragraphs in Bourbaki's Algèbre linéaire [Actualités Sci. Ind., no. 1032, Hermann, Paris, 1947; MR 9, 406]. *J. Dieudonné* (Evanston, Ill.).

Tsuzuku, Tosirô. On a conjecture of Kaplansky on quadratic forms. J. Math. Soc. Japan 6, 325–331 (1954).

Let F be a field of characteristic different from 2 in which -1 is a sum of squares. Let A denote the order of the multiplicative group of F modulo squares. Let B denote the smallest integer such that -1 is a sum of B squares. Let C denote the smallest integer such that every quadratic form in $C+1$ variables is a null form. The reviewer [same J. 5, 200–207 (1953); MR 15, 500] conjectured $C \leqq A$ and proved it for $B \leqq 4$. In the review a proof of the conjecture by M. Kneser is given. Tsuzuku's method gives the sharper result that C is strictly smaller than A for $B > 4$. For instance, $C \leqq 23A/32$ for $B=8$, and $C \leqq 301A/512$ for $B=16$. [Bibliographical note. H. Kneser [Jber. Deutsch. Math. Verein. 44, 143–146 (1934)] proved among other things that B is a multiple of 16 if it is larger than 8. It is not known whether B can actually be larger than 4.] *I. Kaplansky* (Los Angeles, Calif.).

Barbilian, D. Solution exhaustive du problème de Steinitz. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 2, 195–259 (misprinted 189–253) (1951). (Romanian. Russian and French summaries)

If S is a subfield of the commutative field F, then let $G(S)$ be the group of all those automorphisms of F which leave invariant every element in S; and if A is a group of automorphisms of F, then denote by $H(A)$ the subfield of fixed elements of A. The field F is called a Dedekindian extension of its subfield D, if $B = H[G(B)]$ for every field B between D and F; and F is called a Krullian extension of its subfield K, if $U = G[H(U)]$ for every subgroup U of $G(K)$. Extending well known theorems of the Galois theory of algebraic extensions, the author proves that F is a Dedekindian extension of its subfield D if, and only if, M^* is normal and separable over M whenever M is a field between D and F and M^* is the totality of elements in F which are algebraic over M. This implies in particular that Dedekindian extensions are algebraic whenever the characteristic is different from 0. The author shows furthermore that F is both Dedekindian and Krullian over its subfield S if, and only if, F is a finite (algebraic), normal and separable extension of S. [The reader should consult a subsequent presentation of this theory due to Krull, J. Reine Angew. Math. 191, 54–63 (1953); MR 15, 97.] *R. Baer.*

Toyoda, Goro, and Hattori, Akira. On the multiplicative group of simple algebras. J. Math. Soc. Japan 6, 262–265 (1954).

Généralisant un résultat antérieur [A. Hattori, same J. 4, 205–217 (1952); MR 14, 723], les auteurs démontrent le théorème suivant: soit A une algèbre simple de rang fini sur un corps F ayant une infinité d'éléments, et soit B une sous-algèbre de A distincte de F et de A. Considérons l'ensemble des conjuguées $B_t = tBt^{-1}$ de B, où t parcourt l'ensemble des éléments inversibles de A. Alors les éléments inversibles $s \in A$ tels que $sB_ts^{-1} = B_t$ pour tout t inversible dans A, sont exactement les éléments $\neq 0$ de F. La démonstration s'appuie sur le lemme suivant relatif aux produits tensoriels: si B et C sont deux algèbres sur F ($C \neq F$) ayant un élément unité, $A = B \otimes C$ leur produit tensoriel (B et C étant identifiées à des sous-algèbres de A), $t = b + c$ ($b \in B$, $c \in C$, c non-$\in F$) un élément inversible de A, alors $B \cap (tBt^{-1})$ est l'ensemble des éléments de B qui permutent avec b. *J. Dieudonné* (Evanston, Ill.).

Jenner, W. E. Arithmetics of algebras over algebraic function fields of several variables. Portugal. Math. 13, 35–36 (1954).

Let A be an associative algebra with unit element and of finite dimension over the quotient field k of a Noetherian integral domain $\mathfrak{o}$ $(\subset k)$. Using definitions and results of an earlier paper [Compositio Math. 11, 187–203 (1953); MR 16, 7], the author proves: (i) If $\mathfrak{O}$ is an order of A, $\mathfrak{a}$ an integral $\mathfrak{o}$-ideal, and $\mathfrak{a}\mathfrak{O}$ the direct intersection of finitely many block ideals B_i, then the B_i commute. (ii) If $\mathfrak{o}$ is integrally closed in k, the mapping $\mathfrak{a} \rightarrow \mathfrak{a}\mathfrak{O}$ from the set of integral ideals $\mathfrak{a}$ of $\mathfrak{o}$ onto a set of integral ideals of $\mathfrak{O}$ is an

isomorphism with respect to sum and product, and, if the elements of $\mathfrak{O}$ have a unique representation of the form $\sum_{i=1}^{n} \alpha_i \eta_i$ with $\alpha_i \in \mathfrak{o}$, $\eta_i \in \mathfrak{O}$, also with respect to intersection.
A. Jaeger (Cincinnati, Ohio).

Hellman, Morton J. Lie algebras arising from systems of linear differential equations. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-10**, i+12 pp. (1955).

Let $A(t)=\sum_0^\infty A_m t^m$ be a square $n\times n$ matrix whose elements are analytic functions of the real variable t and $A'(t)$ its derivative. "It is known that if (a) the commutator $[A'(t), A(t)]=A'(t)A(t)-A(t)A'(t)=0$, or (b) $[A(s), A(t)]=0$ for all s, t, then the solution of the initial-value problem for the system of linear ordinary differential equations $Y'=A'(t)Y$ can be written as $Y=\exp[A(t)]$. Obviously condition (b) implies (a). In this report examples are investigated in which (a) is satisfied but not (b). This may happen even when $A(t)$ is a matrix of polynomials, and even when $A(t)$ cannot be transformed into triangular form. The present work extends results obtained in a previous paper by W. Magnus [Math. Res. Group, Washington Square Coll. Arts Sci., New York Univ., Res. Rep. No. **BR-3** (1953); MR **15**, 97]." (Author's summary.) Making use of the abbreviation

$$\{B, A_0^{m-1}\}=[[[\cdots[B, A_0]A_0]\cdots]A_0]$$

it is shown first that $[A_m, A_0^{m-1}]=0$ if (a) is satisfied. By means of these formulas it is then proved that there is a constant matrix C such that $C^{-1}A(t)C=B_1(t)+\cdots+B_r(t)$, where the $B_\rho(0)$ $(\rho=1, \cdots, r)$ are J_ρ-rowed square matrices having a single characteristic root (of multiplicity J_ρ) only. For the construction of examples it is assumed that $A(t)$ is a polynomial in t, in which case the A_r generate a nilpotent Lie algebra.
H. Schwerdtfeger (Melbourne).

Epstein, Marvin P. An existence theorem in the algebraic study of homogeneous linear ordinary differential equations. Proc. Amer. Math. Soc. 6, 33–41 (1955).

Let $L(y)$ be an arbitrary homogeneous linear ordinary differential polynomial of order $n\geqq 1$ with coefficients in an ordinary differential field F of characteristic 0 with field of constants C. Kolchin [Bull. Amer. Math. Soc. **54**, 927–932 (1948); MR **10**, 349] proved the existence of a regular system of zeros $(\eta_1, \cdots, \eta_n)$ of $L(y)$, i.e. a fundamental system of zeros such that the field of constants D of $F\langle\eta_1, \cdots, \eta_n\rangle$ is algebraic over C. The author sharpens this result by showing that a regular system can always be chosen so that the corresponding extended field D is normal over C. All elements which are used in the proof are assumed to belong to a universal differential extension of F [Kolchin, Amer. J. Math. **75**, 753–824 (1953); MR **15**, 394]. Two cases are distinguished according to whether or not there exist two regular systems of zeros of $L(y)$ such that the corresponding extended fields of constants are unequal. In the first case the required regular system is constructed by induction on n, in the second case it is shown that the extended field of constants, which is then common to all regular systems, is equal to C or to the algebraic closure $\bar{C}$ of C.
A. Jaeger (Cincinnati, Ohio).

Cohn, Richard M. Finitely generated extensions of difference fields. Proc. Amer. Math. Soc. 6, 3–5 (1955).

Let H, J, K, be difference fields such that $H\subseteq J\subseteq K$. It is proved that if K is a finitely generated extension of H, $K=H\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$, then J is also a finitely generated extension of H.
H. Levi (New York, N. Y.).

Cohn, Richard M. On the intersections of the components of a difference polynomial. Proc. Amer. Math. Soc. 6, 42–45 (1955).

The main result of this note is that all solutions common to two distinct components of the manifold of a difference polynomial annul the separants of the polynomial. The proof is based on the following algebraic lemma. Let K be a field, let $u_1, \cdots, u_q$; $x_1, \cdots, x_p$ be indeterminates and let $F_1, \cdots, F_p$ be polynomials of $K[u_1, \cdots, u_q; x_1, \cdots, x_p]$ such that for each j, $j=1, \cdots, p-1$, F_j is free of x_k, $k>j$. Then any zero of $F_1, \cdots, F_p$ which annuls no $\partial F_j/\partial x_j$ belongs to exactly one component of $\{F_1, \cdots, F_p\}_0$. This component, furthermore, is of dimension q.
H. Levi.

Theory of Groups

Hashimoto, Hiroshi. On the kernel of semigroups. J. Math. Soc. Japan 7, 59–66 (1955).

The paper duplicates results of A. H. Clifford [Amer. J. Math. **70**, 521–526 (1948); MR **10**, 12].
A. D. Wallace.

Tamura, Takayuki, and Kimura, Naoki. On decompositions of a commutative semigroup. Kōdai Math. Sem. Rep. **1954**, 109–112 (1954).

By a decomposition of a semigroup S into a semilattice is meant a congruence relation $\sim$ in S such that $S \bmod \sim$ is a semilattice. In other words, $\sim$ is an equivalence relation in S such that (1) $a\sim b$ and $c\sim d$ imply $ac\sim bd$, (2) $a\sim b$ implies $a\sim ab$, and (3) $ab\sim ba$. The conjunction of all such relations is another such, and thus S admits a greatest (finest) decomposition into a semilattice. [In the proof of this theorem, the authors omit condition (3), and thus obtain the greatest decomposition of S into a band (non-commutative semilattice).] If S is commutative, the greatest decomposition is given by defining $a\sim b$ to mean that each divides some power of the other. Each congruence class S_α thereunder is a semigroup containing at most one idempotent element. If S_α contains an idempotent e, it is inversible ($ax=xa=e$ solvable for x given a). Unipotent inversible semigroups have been studied by Tamura [same Rep. **1954**, 93–95; MR **16**, 443]. A similar decomposition of any commutative semigroup S into a semilattice has been given by Numakura [Proc. Japan Acad. **30**, 262–265 (1954); MR **16**, 214]. An example is given to show that the two decompositions are distinct.
A. H. Clifford.

Artzy, Rafael. A note on the automorphisms of special loops. Riveon Lematematika 8, 81 (1954). (Hebrew. English summary)

The author considers a finite loop L with the property $xy\cdot x'=y$, $xx'=1$, x, $y\in L$. He shows that the mapping T: $x\to x'$ is an automorphism and that the cyclic group C_N generated by T lies in the centre of the automorphism group of L. Here N is the lowest common multiple of the lengths of the "cycles" of L, i.e. of sequences $x_1, x_2, \cdots, x_n$ such that $x_i'=x_{i+1} \bmod n$. He further shows that if an automorphism of L maps a upon b, then a and b belong to cycles of same length.
J. Levitzki (Jerusalem).

Kawada, Yukiyosi. Cohomology in abstract unit groups. Proc. Amer. Math. Soc. 6, 12–15 (1955).

Let G be a finite group, E a G-module, A_1 and A_2 G-submodules of E such that $A_1\cap A_2=(0)$. If the cohomology groups for G in E are (0), the consideration of the various

module extensions involving E, A_1, A_2 and A_1+A_2, and of the corresponding exact sequences connecting the cohomology groups for G in the various modules, gives isomorphisms and exact sequences

$$H^r(G, E/A_1) \approx H^{r+1}(G, A_1) \approx H^{r+1}(G, (A_1+A_2)/A_2),$$

and

$$(0) \to H^r(G, E/A_2) \to H^r(G, E/(A_1+A_2)) \to H^{r+1}(G, (A_1+A_2)/A_2) \to (0).$$

In particular, if G is a subgroup of a finite group K, if E is taken to be the integral group ring of K, and A_i is either $\sum_{\sigma\in K} Z(1-\sigma)$ or $Z\sum_{\sigma\in K}\sigma$, these results give generalizations of the results of Clifford and MacLane [Trans. Amer. Math. Soc. 50, 385–406 (1941); MR 3, 194]. In fact, the recent developments in the cohomology theory for finite groups have removed all the computational difficulties which were involved in the older proofs of such results.

G. Hochschild (Urbana, Ill.).

Frucht, Robert. Remarks on finite groups defined by generating relations. Canad. J. Math. 7, 8–17; corrections, 413 (1955).

The author describes a method for deriving a group of order $2h$ with $k+1$ involutory generators from any group of order h with k generators, which he calls the "duplication principle". He also proves that if $\mathfrak{F}$ and $\mathfrak{G}$ are two groups, each having k-generators such that for $1 \leqq r \leqq k$ the rth generators of $\mathfrak{F}$ and $\mathfrak{G}$ have relatively prime periods, then the direct product $\mathfrak{F}\times\mathfrak{G}$ is generated by k elements. These results are used for finding symmetrical graphs of degree 3: from a group of order n with two generators S_1 and S_2 which admits an automorphism such that the three elements S_1, S_2, $(S_1S_2)^{-1}$ undergo a cyclic permutation a symmetrical graph of degree 3 with $2n$ vertices and $3n$ edges can be derived. Among the graphs derived by these methods is the author's one-regular graph of degree 3 and girth 12 with 432 vertices [same J. 4, 240–247 (1952); MR 13, 857].

G. A. Dirac (Vienna).

Schenkman, Eugene. Two theorems on finitely generated groups. Proc. Amer. Math. Soc. 5, 497–498 (1954).

If $G=\{H, b\}$, where H is finite, b is of finite order, and $[[H, b], b]=1$, then G is finite. If $G=\{b_1, \cdots, b_n\}$, where $[[G, b_i], b_i]=1$, then G is nilpotent of class at most n.

Graham Higman (Oxford).

Schenkman, Eugene. The existence of outer automorphisms of some nilpotent groups of class 2. Proc. Amer. Math. Soc. 6, 6–11 (1955).

As a step towards deciding whether a nilpotent group always has an outer automorphism it is proved (1) that the order of a finite non-abelian group G which is nilpotent of class 2 (i.e. whose centre contains the commutator-group of G) divides the number of automorphisms of G; and (2) a p-group G whose commutator-group is contained in the centre of G and contains the subgroup generated by the p^kth powers (for some k) of the elements of G has an outer automorphism.

H. A. Thurston (Bristol).

Plotkin, B. I. On the theory of solvable groups without torsion. Mat. Sb. N.S. 36(78), 31–38 (1955). (Russian)

The author calls a group solvable when it is solvable in the generalized sense of possessing an infinite ascending solvable series; thus he distinguishes between invariantly solvable and normally solvable. A group having a finite rational series is called ρ-finite. A locally normally solvable R^{**}-group every abelian subgroup of which has finite rank is the extension of its maximal nilpotent invariant subgroup with finite rational rank by means of an abelian torsion-free group; it is furthermore solvable and locally ρ-finite. In a locally normally solvable R^{**}-group G these four conditions are equivalent: (1) G is ρ-finite; (2) G has finite special Mal'cev rank; (3) G satisfies the minimal condition for isolated subgroups; (4) G satisfies the maximal condition for isolated subgroups. In an R-group with an invariant rational series each of the two conditions is sufficient that the series be central: (1) every factor of the series is an infinite cyclic group; (2) no factor of the series is a cyclic group. An R-group with an ascending invariant rational series is the extension of its maximal normal subgroup with ascending central series by means of an abelian torsion-free group every rational subgroup of which is cyclic. If a group is an extension of a locally nilpotent normal subgroup by an abelian group, then the set of nil-elements in the group coincides with the maximal locally nilpotent normal subgroup.

R. A. Good (College Park, Md.).

Baer, Reinhold. Supersoluble groups. Proc. Amer. Math. Soc. 6, 16–32 (1955).

A group is supersoluble if every homomorphic image $\neq 1$ has a cyclic normal subgroup $\neq 1$. Such groups form a natural generalisation of upper nilpotent groups, and share many of their properties. Thus finitely many elements of odd order in a supersoluble group generate a subgroup of odd order, whence the set of elements of odd order (though not, in general, the set of elements of finite order) form a subgroup. If G is supersoluble then the derived group G' is upper nilpotent; so is the group G^2 generated the squares of elements of G, if it has no elements of finite order. If G^2 has elements of finite order, they form a subgroup. A finitely generated supersoluble group satisfies the maximal condition for subgroups, and possesses an upper nilpotent subgroup of finite index.

Graham Higman (Oxford).

Gaschütz, Wolfgang. Gruppen, deren sämtliche Untergruppen Zentralisatoren sind. Arch. Math. 6, 5–8 (1954).

The author shews that a finite group has the property indicated in the title if and only if it is a direct product of generalised dihedral groups of coprime orders, where a generalised dihedral group means a non-abelian group of order pq, p and q being primes. The main weapon is the determination by Suzuki [Trans. Amer. Math. Soc. 70, 345–371 (1951); MR 12, 586] of all finite soluble groups the lattice of whose subgroups has an involutory dual automorphism.

Graham Higman (Oxford).

Gaschütz, Wolfgang. Endliche Gruppen mit treuen absolut-irreduziblen Darstellungen. Math. Nachr. 12, 253–255 (1954).

The necessary and sufficient condition that a finite group $\mathfrak{G}$ have a faithful irreducible representation was first given by Shoda [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 2, 51–72 (1930)] and later simplified by Weisner [Amer. J. Math. 61, 709–712 (1939); MR 1, 6] and Kochendörffer [Math. Nachr. 1, 25–39 (1948); MR 10, 281]. In the present formulation the author calls the direct product

$$\mathfrak{S}=\mathfrak{M}_1\times\mathfrak{M}_2\times\cdots\times\mathfrak{M}_t$$

of the minimal normal subgroups $\mathfrak{M}_i$ of $\mathfrak{G}$ the 'base' of $\mathfrak{G}$, and writes $\mathfrak{S}=\mathfrak{A}\times\mathfrak{H}$, where $\mathfrak{A}$ is abelian and $\mathfrak{H}$ contains no normal abelian subgroup. The condition is as follows: a

finite group $\mathfrak{G}$ has a faithful irreducible representation in an algebraically closed field of characteristic zero if and only if the base $\mathfrak{S}$ (or alternatively, $\mathfrak{A}$) of $\mathfrak{G}$ is generated by a single class of conjugates in $\mathfrak{G}$. The proof is based on an elegant application of the exclusion principle.

G. de B. Robinson (Toronto, Ont.).

Ado, I. D. On the theory of linear representations of finite groups. Mat. Sb. N.S. **36**(78), 25–30 (1955). (Russian)

If A is an automorphism of the finite group G, then G is partitioned into A-classes by the concept: an element a_1 is A-conjugate to the element a provided $a_1 = x^{-A}ax$ for some $x \in G$. If Γ is a linear representation of G, then Γ^A denotes the representation that maps $y \in G$ onto the matrix which corresponds under Γ to y^A. The number of A-classes in G equals the number of absolutely irreducible linear representations of G that remain equivalent to themselves under A.

R. A. Good (College Park, Md.).

Berman, S. D. On representations of a semi-direct product of abelian groups. Dokl. Akad. Nauk SSSR (N.S.) **98**, 177–180 (1954). (Russian)

Let a finite group G be a semi-direct product of two abelian subgroups H_1 and H_2 (i.e. $G = H_1H_2$, $H_1 \cap H_2 = 1$ and H_1 is a normal subgroup of G). Denote by $R(G, K)$ the group algebra of G over the complex field K and by e_i $(i=1, \cdots, n)$, respectively e_i' $(i=1, \cdots, m)$, a complete orthogonal set of minimal idempotents of the subalgebra $R(H_1, K)$, respectively $R(H_2, K)$. Every element b of H_2 induces an automorphism $a \to b^{-1}ab$ of the subalgebra $R(H_1, K)$ and a permutation of its minimal idempotents. Two idempotents e_i, e_j belong to the same domain of transitivity if there is an automorphism which maps e_i and e_j. Thus the e_i are split into mutually disjoint domains of transitivity $(e_1^{(j)}, \cdots, e_{\beta_j}^{(j)})$ $(j=1, \cdots, r)$. Denote by F_i' the subgroup of H_2 consisting of all elements b such that $be_1^{(i)} = e_1^{(i)}b$, and let $u_1^{(i)}, u_2^{(i)}, \cdots, u_{m_i}^{(i)}$ denote a complete orthogonal set of minimal idempotents in the algebra $R(F_i', K)$. Now set $\bar{e}_i = \sum_{s=1}^{\beta_i} e_s^{(i)}$, $\bar{e}_{ij} = \bar{e}_ie_j'$ and $e_{is}'' = \bar{e}_iu_s^{(i)}$ $(i=1, \cdots, t;\ j=1, \cdots, m;\ s=1, \cdots, m_i')$. The author shows that the $\bar{e}_{ij}$ (respectively e_{is}'') form a complete orthogonal set of minimal (respectively minimal central) idempotents of $R(G, K)$. He also formulates a necessary and sufficient condition for any two $\bar{e}$'s to belong to the same (absolutely) irreducible representation of G. In case H_2 is a cyclic group he also determines the degrees of the irreducible representations of G.

J. Levitzki (Jerusalem).

Ehrenfeucht, A. On a certain problem of K. Kuratowski and A. Mostowski in the theory of groups. Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 471–473 (1955).

Let us denote by C the set of all integers, by c the set of all its subsets, by C^c the additive group of all functions μ: $c \to C$, and by $\mathfrak{M}$ the subgroup of all μ such that $\mu(X_1+X_2) = \mu(X_1)+\mu(X_2)$ for $X_1 \cdot X_2 = 0$ $(X_1, X_2 \in c)$. The author shows that $\mathfrak{M}$ is isomorphic with C^c, thus giving an affirmative answer to a question raised by C. Kuratowski and A. Mostowski [Colloq. Math. **2**, 212–215 (1952); MR **14**, 131].

J. Tits (Brussels).

Edge, W. L. The isomorphism between $LF(2, 3^2)$ and $\mathfrak{A}_6$. J. London Math. Soc. **30**, 172–185 (1955).

L'isomorphisme (classique) entre les deux groupes indiqués dans le titre est obtenu ici comme conséquence d'une étude approfondie de la géométrie de la droite projective sur le corps J à 9 éléments. L'auteur remarque d'abord que si 4 points de la droite forment une division harmonique quand on les prend dans un certain ordre, il en est encore ainsi quand on les permute arbitrairement. Il décrit ensuite un procédé intrinsèque pour associer en "quintuples" les diverses divisions harmoniques, de sorte que dans chaque quintuple de division harmoniques, un point de la droite apparaisse exactement 2 fois. Ces quintuples sont eux-mêmes au nombre de 12, et se divisent en deux classes d'intransitivité pour le groupe projectif; il est facile alors de voir que ce groupe, considéré comme groupe de permutations d'une de ces classes, s'identifie au groupe alterné sur 6 objets. L'auteur étudie aussi les formes hermitiennes à 2 variables sur J et montre que les droites où elles s'annulent correspondent exactement aux points d'une division harmonique. Enfin, il décrit sommairement l'isomorphisme classique entre le groupe projectif de la droite sur J et le "second groupe orthogonal projectif" à 2 variables sur le corps à 3 éléments.

J. Dieudonné (Evanston, Ill.).

de Groot, J., and Dekker, T. Free subgroups of the orthogonal group. Compositio Math. **12**, 134–136 (1954).

Les auteurs démontrent que dans le groupe des rotations de l'espace euclidien à 3 dimensions, il y a un sous-groupe libre ayant un ensemble de générateurs qui a la puissance du continu. On sait qu'il existe de tels sous-groupes libres à 2 générateurs, et par suite (en raison d'un théorème sur les groupes libres) un sous-groupe libre G_0 à $\aleph_0$ générateurs. Partant de G_0, les auteurs construisent alors le groupe cherché par un raisonnement de récurrence transfinie.

J. Dieudonné (Evanston, Ill.).

Pais, A. Spherical spinors in a Euclidean 4-space. Proc. Nat. Acad. Sci. U. S. A. **40**, 835–841 (1954).

The author's theory of baryons [same Proc. **40**, 484–492 (1954); MR **16**, 320] requires a study of properties of the rotation group in Euclidean four-space. He here obtains a set of functions related to this group as the half-integer spherical harmonics are to the three-dimensional rotation group. These he defines as solutions of a certain first-order linear differential equation involving an invariant operator formed from Dirac matrices and angular momentum operators. He asserts, without proof, that the solutions which he obtains constitute a complete set. Preparatory to the solution of this equation is a section on hyperspherical functions containing results which are mostly known; for example, his identities (12) and (13) are equations (1′), p. 85 of the survey article on the representations of the three-dimensional rotation group by Gelfand and Šapiro [Uspehi Mat. Nauk (N.S.) **7**, no. 1647, 3–117 (1952); MR **13**, 911] (This article, to which Pais does not refer, is of great interest to physicists but is not yet widely known.)

A. J. Coleman (Toronto, Ont.).

Chevalley, C. On algebraic group varieties. J. Math. Soc. Japan **6**, 303–324 (1954).

Let G be an irreducible r-dimensional algebraic group of linear automorphisms of a finite-dimensional vector space V over a field K of characteristic 0. The main results are as follows: (1) Suppose that the Lie algebra of G has a Cartan subalgebra, the characteristic polynomials of whose elements (regarded as linear transformations of V) split into linear factors with coefficients in K. Then the field of the rational functions on G is a purely transcendental extension of K. (2) The field of the rational functions on G is always contained in a purely transcendental extension of transcendence degree r of K.

The second result (but not the first) was proved by E. Picard in the case where K is the complex number field. In the general case, it is here shown by means of an example that the field of rational functions on G is not always purely transcendental over K (but note that, by (1), this is the case whenever K is algebraically closed).

The proofs depend heavily on the author's theory of algebraic groups and Lie algebras [Théorie des groupes de Lie, vols. II and III, Hermann, Paris, 1951, 1955; MR **14**, 448]. In particular, by using the structure theory for algebraic Lie algebras, (1) is reduced eventually to the extreme cases where either the Lie algebra of G consists only of nilpotent linear transformations, or G is abelian and V is semisimple for G. The proof of (2) involves considerable additional difficulties. It is shown that those rational functions on G which are constant on every Cartan subgroup constitute a purely transcendental field extension L of K, and that the whole field of rational functions on G is isomorphic with the field of rational functions on some Cartan subgroup of the group G^L obtained from G by enlarging the base field to L. This Cartan subgroup is then analysed by first making a finite Galois extension of L so as to secure the condition of (1) and then leading the result back to the original group by considering the operations of the Galois group on the extended algebraic group.

G. Hochschild (Urbana, Ill.).

Freudenthal, Hans. Zur Berechnung der Charaktere der halbeinfachen Lieschen Gruppen. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 369–376, 487–491 (1954).

Let L be a semi-simple Lie algebra over the complex numbers. For a given irreducible representation of L, if h is a weight, let m_h be the dimension of the corresponding weight space. The Casimir operator commutes with every operator in the representation, consequently it has the form $s\cdot I$, where I is the identity operator. The "Hauptformel" of the first note is the recursion $m_h s = m_h(h, h) + \sum m_g(g, \alpha)$ where the sum is taken over all pairs α and g such that α is a root and g a weight of the form $h+p\alpha$, p a positive integer. Since $s=(h, h)$ only for the highest weight, in which case $m_h=1$, this formula determines all other values of m_h. There follows an algebraic proof of Weyl's formula for the character of the representation.

In the second note, the Hauptformel is applied to explicitly compute the dimensions of the various weight spaces for all representations of E_8 for which $s \leq 300$, the latter being the value of s appearing in the representation whose highest weight is the eighth fundamental weight. The facts about these 27 representations are presented in six tables. Numerous checks were applied to insure the accuracy of the computation.

A. M. Gleason.

Dynkin, E. B. Topological characteristics of homomorphisms of compact Lie groups. Mat. Sb. N.S. **35**(77), 129–173 (1954). (Russian)

In earlier papers the author has studied the homomorphisms of the homology algebras of a compact Lie group G into those of another such group G' induced by a homomorphism F of G into G'. Such a homomorphism determines a subgroup $F(G)$ of G', and the theme of present paper is the related connection between the homology of a compact Lie group and that of the subgroups of the group. The results are useful for treating the homology of homogeneous spaces and can be used to derive various known results, notably a theorem of Samelson [Ann. of Math. (2) **42**, 1091–1137 (1941); MR **3**, 143].

The paper studies in particular the known correspondence between polynomials on the maximal abelian subalgebras of the Lie algebra of the group that are invariant under the Weyl group and elements of the homology algebra of the group. The topological characters of representations are shown to be expressible in terms of systems of weight vectors of the representations. The special case when G' is the unitary group is stressed and a formula is derived for this case expressing the image of a primitive element x of the homology algebra of G in terms of the highest weight of the unitary representation, a polynomial x invariant under the Weyl group, a certain element of the Lie algebra that is independent of F, and a canonical set of primitive elements for the homology algebra of G.

I. E. Segal.

NUMBER THEORY

Nov 6

***Hardy, G. H., and Wright, E. M. An introduction to the theory of numbers. 3rd ed.** Oxford, at the Clarendon Press, 1954. xvi+419 pp. $6.75.

The 3rd edition of this well known standard introduction differs from the 2nd edition (1945) mainly in the addition of some new material embodying advancements in recent years. As the second author writes in his preface, he has refrained from a more thorough revision, for fear of disturbing Hardy's elegant style.

The most important addition is an elementary proof of the prime-number theorem, in Wright's modification [Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 257–267 (1952); MR **14**, 137] of Selberg's version. It replaces complicated operations with sums by the corresponding operations with integrals, thus losing unessentially in elementarity, but gaining considerably in legibility. It fits extremely well in the general atmosphere of the book.

Further additions are, among others: Hermite's short proof for the irrationality of e^y (y rational), and Niven's proof [Bull. Amer. Math. Soc. **53**, 509 (1947); MR **9**, 10] for the irrationality of π, arranged here, without additional trouble, so as to include the case of π^2 (ch. 4); a section on tests for primality, and recent results on special cases (ch. 6); some material about the Prouhet-Tarry problem (ch. 21); an additional section to the chapter on the geometry of numbers (ch. 24); a postscript on prime pairs.

Needless to say that the valuable collection of notes and references has been kept up to date.

N. G. de Bruijn.

***Jones, Burton W. The theory of numbers.** Rinehart & Company, Inc., New York, 1955. xi+143 pp. $3.75.

This book gives a very readable account of the elementary theory of numbers. Its six chapters are devoted to the following topics: 1. The development of the number system; here a brief description of the elementary properties of integers, real numbers (including decimals) and complex numbers is given, Euclid's algorithm is set up and the fundamental theorem of the theory of numbers is deduced. 2. Repeating decimals and congruences; this includes Euler's function $\phi(n)$, perfect numbers and Fermat numbers. 3. Diophantine equations; this includes linear equations, Pell's equation and Pythagorean numbers. 4. Continued

fractions; this includes Fibonacci numbers, expansions of quadratic surds and applications to Pell's equation. 5. Nonlinear congruences; here polynomial congruences are considered and the existence of primitive roots is established. 6. Quadratic residues; this includes Legendre and Jacobi symbols, the reciprocity law and sums of two squares. At each stage numerous exercises for the reader are given illustrating the previous work and pointing towards further developments. Many of these are of a stimulating and even amusing character and are admirably suited to the author's objective of making students think things out for themselves.

R. A. Rankin (Glasgow).

Sanielevici, S. Sur les formes x^2+Ay^2. Acad Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 337–347 (1953). (Romanian. Russian and French summaries)

The author employs an argument due to Hermite to show that if a solution of the congruence $z^2\equiv -A \pmod n$ is known the continued fraction for n/z yields a solution (x, y) of the diophantine equation $x^2+Ay^2=kn$ for some $k<A$. A study is made of various A's, especially $A=7, 11, 19, 67, 163$ for which it is shown that $4p=x^2+Ay^2$ for every prime p that has $-A$ for a quadratic residue. *D. H. Lehmer.*

Hanneken, C. B. Irreducible quintic congruences. Duke Math. J. **22**, 107–118 (1955).

Here those quintic congruences

$$Q_\lambda(x)=x^5+\alpha x^4+\beta x^3+\gamma x^2+\delta x+\epsilon\equiv 0 \pmod{p^n}$$

are studied which belong to an arbitrary finite field $GF(p^n)$ under the group of linear fractional transformations $T: x=(ax'+b)/(cx'+d)$ with coefficients belonging to the same field. Two quintics $Q_\lambda(x)$ and $Q_\theta(x)$ are called conjugate, when they can be transformed into each other by the mentioned transformation. The irreducible quintics are classified in classes of conjugate quintics, the number of such classes and the number of conjugate quintics in a class are determined. To each conjugate set of irreducible quintics over $GF(p^n)$ corresponds a set of five cubic curves in a finite projective three space over $GF(p^n)$, which set determines the first mentioned set. *H. Bergström* (Göteborg).

Lindgren, H. From necklaces to number theorems. Math. Gaz. **39**, 13–19 (1955).

The problem of forming a necklace of n beads so that no two adjacent beads have the same color is shown to lead to the well-known congruence (1) $\sum_{d|n}\mu(d)N^{n/d}\equiv 0 \pmod n$ [for references see Dickson's History of the theory of numbers, v. 1, Carnegie Inst. of Washington, 1919, pp. 84–86]. It is also proved that (2) $\sum at\equiv 0 \pmod{\phi(n)}$ if and only if (3) $\sum aN^t\equiv 0 \pmod n$ for all N. Here t runs through monomials in the prime divisors of n. The author remarks that (3) must be 'an algebraic relation, not merely a fortuitous numerical one'. (Presumably this means that (2) hold for all n, in which case (3) is an immediate consequence of (1). In this connection it may be of interest to recall the theorem of Gegenbauer [Monatsh. Math. Phys. **11**, 287–288 (1900)] that if $\sum f(d)\equiv 0 \pmod n$, then $\sum f(d)a^{n/d}\equiv 0 \pmod n$ for all a).

The paper also contains some discussion of the more difficult problems in which three consecutive beads are to have different colors. *L. Carlitz* (Durham, N. C.).

Bang, Thøger. Congruence properties of Tchebycheff polynomials. Math. Scand. **2**, 327–333 (1954).

The polynomial $T_n(x)$ is defined by means of $T_n(x)=\cos n$ (arc cos x). Let p denote a fixed odd prime. If x is an integer, the rank $r=r_x$ is defined as the least positive r such that $T_r(x)\equiv 1 \pmod p$; it is proved that r exists for all x. The main results of the paper are contained in the following theorem. There exists a one-to-one correspondence between the residue classes (mod p) and the p distinct values of $\cos(2\pi h/(p+1))$ and $\cos(2\pi k/(p-1))$, where h and k are arbitrary integers, such that when x corresponds to $\alpha=\cos\xi$ then $T_j(x)$ corresponds to $T_j(\alpha)=\cos j\alpha$ for every j. The rank of x is equal to the least positive denominator d, where $\xi=2\pi n/d$.

A necessary and sufficient condition that $M=2^n-1, n>2$, be a prime is that $M|T_{2^{n-2}}(x)$, where $x=2$ or any $T_j(2)$ with j odd. *L. Carlitz* (Durham, N. C.).

Selmer, Ernst S. The diophantine equation

$$ax^3+by^3+cz^3=0.$$

Completion of the tables. Acta Math. **92**, 191–197 (1954).

The author continues his studies [Acta Math. **85**, 203–362 (1951); MR **13**, 13] of the equations (1) $x^3+y^3=Az^3$ and (2) $ax^3+by^3+cz^3=0$. The tables of basic solutions of equation (2) are completed for all positive cube-free $abc\leqq 500$, with extensions beyond 500 in some cases. Part of this was done by work with an electronic computer. These tables, together with cases proved unsolvable by the method of J. W. S. Cassels [ibid. **82**, 243–273 (1950); MR **12**, 11] enable the writer to complete the solutions of (1) for all positive cube-free $A\leqq 500$. *I. Niven* (Eugene, Ore.).

Wilson, Neil Y. Conjectures as to a factor of $2^p\pm 1$. Edinburgh Math. Notes no. **39**, 6–9 (1954).

The author gives seven conjectured theorems giving necessary and sufficient conditions for $2^n\pm 1$ to be divisible by $p=kn+1$ for $k=3, 4, 5, 6, 8, 12, 24$. Except for the case $k=5$, the theorems are unnecessarily complicated restatements of the known criteria for 2 to be a kth power residue of p. The conjecture in the case $k=5$ is very different from the others and is even incorrect since it does not account for the fact that 2 is a quintic residue of $p=3181$, as a little more experimentation would have shown. Actually the author claims that his conjecture holds for $p<5000$. [For a criterion for 2 to be a quintic residue of p see E. Lehmer, Duke Math. J. **18**, 11–18 (1951); MR **12**, 677.]

D. H. Lehmer (Berkeley, Calif.).

Araujo, Roberto. Fermat's theorem for even exponent. Rev. Acad. Ci. Zaragoza (2) **8**, no. 2, 21–24 (1953). (Spanish)

The paper, in two parts, contains arguments purporting to show that $x^4+y^4=z^4$ and $x^{2n}+y^{2n}=z^{2n}$ (n, odd >1) have no solutions in integers $\neq 0$. The first part of the argument does not use Fermat's celebrated descent method but arrives at the conclusion by omitting one case. The second part is difficult to follow because of many typographical errors. The reviewer succeeded in finding his way to equation (7) where omission of a term vitiates the rest of the argument.

D. H. Lehmer (Berkeley, Calif.).

Sanielevici, S. La décomposition d'un nombre entier en une somme de deux carrés. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 5–18 (1953). (Romanian. Russian and French summaries)

The author traces the well known connection between the representation $m=a^2+b^2$ of a number m dividing $1+c^2$ and the continued fraction for m/c. This result is used to study

quadratic equations whose roots have periodic continued fractions with "metasymmetric" partial quotients. The connection with the Pell equation $x^2-Dy^2=\pm 4$ is noted.

D. H. Lehmer (Berkeley, Calif.).

Duparc, H. J. A. Periodicity properties of recurring sequences. II. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indag. Math. 16, 473–485 (1954).

This is the second of two papers [for part I see same Proc. 57, 331–342 (1954); MR 16, 113] which give a redevelopment of the author's dissertation [Univ. of Amsterdam, 1953; MR 15, 200]. It is concerned with divisibility properties of the periods obtained when a linear recurring series is taken with respect to the quite general modulus described in detail in the first paper. The paper also contains a detailed discussion of second, third, and fourth order recurring series as they become special cases of his general theory.

D. H. Lehmer (Berkeley, Calif.).

Ward, Morgan. On the number of vanishing terms in an integral cubic recurrence. Amer. Math. Monthly 62, 155–160 (1955).

Let (T): $T_0, T_1, T_2, \cdots$ be an integral cubic recurrence; that is, the initial values T_0, T_1, T_2 are integers and $T_{n+3}=PT_{n+2}-QT_{n+1}+RT_n$, where P, Q, R are fixed integers and $R\neq 0$. The polynomial $f(z)=z^3-Pz^2+Qz-R$ and the recurrence (T) are said to be associated; also both (T) and $f(z)$ are degenerate or non-degenerate according as the ratio of any pair of different roots of $f(z)$ is, or is not, a root of unity. By a result of Mahler [Nederl. Akad. Wetensch. 38, 50–60 (1935)], if (T) is non-degenerate, $|T_n|$ tends to infinity with n; hence the total number of zeros of any non-degenerate (T) is finite. The present paper contains a proof of the following theorem. If the associated polynomial of (T) is non-degenerate and has integral roots which are co-prime in pairs, then at most three terms of (T) can vanish.

L. Carlitz (Durham, N. C.).

Stolt, Bengt. Über den kleinsten positiven quadratischen Nichtrest. Math. Scand. 2, 187–192 (1954).

The author proves that if p is any prime of the form $8m+5$, different from 5, 13, 37, 61, 109, then the least positive odd quadratic non-residue of p is less than $(p/2)^{1/2}$. This improves by a factor of $2^{-1/2}$ the upper limit $p^{1/2}$ set by a recent result of Rédei [Acta Sci. Math. Szeged 15, 12–19 (1953); MR 15, 102] at least for the case $p\equiv 5$ (mod 8). The author states that a similar result can be proved for primes p of the form $8m+3$, 7. To avoid transient situations the cases $(p/2)^{1/2}<5$ (i.e. $p<50$) are taken up separately. For $e=[(p/2)^{1/2}]\geqq 5$ the author separates 24 cases, namely the residue classes (mod 24). In each case he exhibits two integers a, b, one of which is odd and neither of which exceeds e in absolute value. Both are of the form $(p-3uv)/8$, where u and v are positive odd integers not exceeding e. It follows that one of 3, a, b, u, v is an odd non-residue of p. However, in 12 of the 24 cases certain exceptional e's are encountered. There are less than 100 and the corresponding primes are examined separately. All but 61 and 109 have small odd quadratic non-residues. [The author might have referred to a paper by H. J. Kanold, J. Reine Angew. Math. 188, 74–77 (1950); MR 12, 483.]

D. H. Lehmer.

Skolem, Th. On the least odd positive quadratic nonresidue modulo p. Norske Vid. Selsk. Forh., Trondheim 27, no. 20, 7 pp. (1954).

Let p be any odd prime not of the form $8k+1$. It is established that there exists a constant c such that every p has a positive odd quadratic non-residue not exceeding $cp^{2/5}$. Admissible values of c are computed in the three cases $p\equiv 3, 5, 7$ (mod 8). For example, $c=6.74$ will suffice in case $p\equiv 7$ (mod 8). A short bibliography of earlier work on this topic is to be found in the paper reviewed above.

I. Niven (Eugene, Ore.).

Ricci, Giovanni. Sull'andamento della differenza di numeri primi consecutivi. Riv. Mat. Univ. Parma 5, 3–54 (1954).

The present memoir is in part a sequel to two earlier papers by the same author [Univ. e Politec. Torino. Rend. Sem. Mat. 11, 149–200 (1952); Ann. Scuola Norm. Sup. Pisa (3) 7, 133–151 (1953); MR 14, 727; 15, 202]. The author continues his extensive investigations of various problems concerned with the difference $p_{n+1}-p_n$ of two consecutive primes. Given the interval $(1-\delta)\xi<p_n\leqq\xi$ of length $\delta\xi$, he develops methods for estimating the number N of those differences $p_{n+1}-p_n$ within the interval which do not exceed a specified value. He also studies how to select the bounds of the given interval so that N lies within prescribed limits. In particular, he determines the bounds as a function of p_n so that N is of the same order of magnitude as $\delta\xi/\log\xi$. An interesting theorem states that for δ fixed, $0<\delta<1$, and ξ sufficiently large, at least 55 out of a thousand of the differences $p_{n+1}-p_n$ within the interval $(1-\delta)\xi<p_n\leqq\xi$ satisfy the inequality $p_{n+1}-p_n<\log p_n$. Most of the results, however, are too complicated to be stated briefly.

A. L. Whiteman (Los Angeles, Calif.).

Schinzel, A., et Sierpiński, W. Sur quelques propriétés des fonctions $\varphi(n)$ et $\sigma(n)$. Bull. Acad. Polon. Sci. Cl. III. 2 (1954), 463–466 (1955).

Elementary proofs are given for the following results on the Euler ϕ-function and the sum of the divisors function $\sigma(n)$. Given any positive m, there exist integers n and h such that $\phi(n-1)/\phi(n)>m$, $\phi(n+1)/\phi(n)>m$, $\phi(h)/\phi(h-1)>m$, $\phi(h)/\phi(h+1)>m$. Analogous results are obtained for $\sigma(n)$ by use of a lemma that $\frac{1}{2}n^2<\phi(n)\sigma(n)<n^2$ for all $n>1$.

I. Niven (Eugene, Ore.).

Schinzel, A. Quelques théorèmes sur les fonctions $\varphi(n)$ et $\sigma(n)$. Bull. Acad. Polon. Sci. Cl. III. 2 (1954), 467–469 (1955).

Given any positive integers m and k, it is established that there exist integers n and h such that $\phi(n+i)/\phi(n+i-1)>m$ and $\phi(h+i-1)/\phi(h+i)>m$ for $i=1, 2, \cdots, k$. Analogous results hold for the function $\sigma(n)$ by application of the lemma cited in the previous review.

I. Niven.

LeVeque, W. J. The distribution of values of multiplicative functions. Michigan Math. J. 2 (1953–54), 179–192 (1955).

The author investigates the asymptotic behaviour, as $x\to\infty$ for fixed k, of $T_k(x)$ and $R_k(x)$; here $T_k(x)$ denotes the number of positive integers $\leqq x$ with exactly k divisors, and $R_k(x)$ denotes the number of positive integers $\leqq x$ with exactly k representations as a sum of two squares. If $k=p_1^{\nu_1}\cdots p_s^{\nu_s}$, where $p_1<p_2<\cdots<p_s$, it is shown that

$$T_k(x)\sim\frac{Ax^\alpha(\log\log x)^{\nu_1-1}}{(\nu_1-1)!\log x},$$

where $\alpha=(p_1-1)^{-1}$ and A is a positive constant depending on k which is expressed as a multiple sum. The proof is analytical, and is based on the idea of expressing the relevant generating functions as (essentially) polynomials

in $\log \zeta(s)$, $\log \zeta(2s)$, $\cdots$. The asymptotic expressions that are obtained for $R_k(x)$ take different forms according as k is (a) even, (b) odd and not divisible by 3, (c) odd and divisible by 3. They disprove a conjecture of P. Lévy [Atti Accad. Sci. Torino **75**, 177–183 (1939); MR **1**, 201].

H. Davenport (London).

Selberg, Atle. Note on a paper by L. G. Sathe. J. Indian Math. Soc. (N.S.) **18**, 83–87 (1954).

The author obtains short and simple proofs of the results of Sathe [same J. (N.S.) **17**, 83–141 (1953); **18**, 27–42, 43–61 (1954); MR **15**, 401; **16**, 221] on $\pi_k(x)$, the number of positive integers $\leqq x$ containing exactly k prime factors.

S. Chowla (Boulder, Colo.).

Ritson, Max. De-gaussing Gauss. Math. Gaz. **39**, 45–46 (1955).

The author points out that a previous attempt by B. E. Lawrence [Math. Gaz. **37**, 280 (1953)] to improve Gauss' approximation $x/\log x$ to the number $\pi(x)$ of primes $\leqq x$ for small x leads to a formula that is asymptotically incorrect. He offers as a substitute

$$\log \pi(x) = \frac{\pi}{10} x^f \{1 - x \log(1 + x^{-1})\},$$

where f is a function of x which "decreases slowly and steadily to a limit slightly less than 1". Without specifying this function or justifying the factor $\pi/10$ the author gives data to show that the fit is good for $x \leqq 100$ and for $x = 1000$.

D. H. Lehmer (Berkeley, Calif.).

Skewes, S. On the difference $\pi(x) - \mathrm{li}\, x$. II. Proc. London Math. Soc. (3) **5**, 48–70 (1955).

In Part I of this paper the author assumes hypothesis (H): Every complex zero $\rho = \beta + i\gamma$ of Riemann's zeta function $\zeta(s)$ satisfies

$$\beta - \tfrac{1}{2} \leqq X_1^{-3} \log^{-2} X_1 \quad \text{provided} \quad |\gamma| < X_1^3.$$

On this assumption he proves that $\pi(x) - \mathrm{li}\, x > 0$ for some x satisfying $2 \leqq x < X_1 = \exp\exp\exp(7.703)$.

In a previous paper [J. London Math. Soc. **8**, 277–283 (1933)] the author has obtained a larger value of X_1 (the so-called Skewes number) on the assumption of the Riemann hypothesis, whose truth would, of course, imply hypothesis (H). In Part II of the paper the author assumes (NH), the negation of (H). Assuming (NH) the author proves that $\pi(x) - \mathrm{li}\, x > 0$ for some $x < X_2$ where

$$X_2 = \exp_{10} \exp_{10} \exp_{10} \exp_{10} 3.$$

The proofs are complete, except for the omission of certain computational details.

S. Chowla (Boulder, Colo.).

Bambah, R. P. Four squares and a kth power. Quart. J. Math., Oxford Ser. (2) **5**, 191–202 (1954).

The author proves the following theorem: Let $\lambda_1, \cdots, \lambda_4, \mu$ be non-zero real numbers, not all of the same sign and such that at least one ratio λ_r/λ_s is irrational; let k be a positive integer. Then there exist arbitrarily large P such that the inequalities $1 \leqq x_r \leqq P^k$, $r = 1, \cdots, 4$; $1 \leqq x_5 \leqq P^2$; $|F(x)| = |\sum_{r=1}^{4} \lambda_r x_r^2 + \mu x_5^k| < 1$ have more than γP^{2k+2} solutions, where γ is a positive number depending only on λ_r and μ. Similar results were obtained by Davenport and Heilbronn [J. London Math. Soc. **21**, 185–193 (1946); MR **8**, 565] and by Watson [Proc. London Math. Soc. (3) **3**, 170–181 (1953); MR **15**, 291]. The method of proof is that of Davenport and Heilbronn with some modification due to Watson. It consists of representing the number of solutions of the stated inequalities as integrals involving exponential sums, and then replacing the sums by integrals and estimating the error by the usual Hardy-Littlewood method.

R. D. James (East Lansing, Mass.).

***Kuhn, P. Neue Abschätzungen auf Grund der Viggo Brunschen Siebmethode.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 160–168 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The author presents an interesting method for sharpening results obtained by the sieve method. The idea is a simple one, but the results are surprising. It is based on the observation that if an integer has $v+w+1$ or more prime factors, then at least $w+1$ of them cannot be too large. The details are given for one example and the following theorem is proved. For sufficiently large x there is always an integer between $x - x^{1/v}$ and x that has at most $v+w$ prime factors. Here v is any positive integer and w is the least integer satisfying the inequality $\log(6v - w) \leqq 0.968(w+1)$. Two numerical examples are $v=3$, $w=2$ and $v=100$, $w=6$. The author indicates that from the work of Buchstab [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **29**, 544–548 (1940); MR **2**, 348] and Tartakowski [ibid. **23**, 126–129 (1939)] it follows that the same result holds if w is the least positive integer satisfying $\log 5(6v - w) - \log(w+6) \leqq 1.097(w+1)$. For example, $v=2$, $w=1$ and $v=100$, $w=5$. Other results which have been obtained by the sieve method may also be improved. For example, it can be shown that there are infinitely many integers of the form x^2+1 which have at most 3 prime factors.

R. D. James (East Lansing, Mass.).

Gupta, Hansraj. On a generating function in partition theory. Proc. Nat. Inst. Sci. India **20**, 582–586 (1954).

Let

$$f(x, r) = r!/(1-x)(1-x^2)\cdots(1-x^r) = r! \sum p(n, r) x^n,$$

where $p(n, r)$ is the number of partitions of n into at most r parts. Writing $f(x, r) = \sum_{j \geqq 0} A_j(r)(1-x)^{j-r}$, the author shows that $A_j(r) = \sum_{i=1}^{2j} q_{i-2}(j)\binom{r}{i}$, where the $q_i(j)$ are independent of r, and are expressible in terms of $A_0(i')$, $\cdots$, $A_{j-1}(i')$, $2 \leqq i' \leqq i+2$. In this way the functions $A_j(r)$ are computed for $0 \leqq j \leqq 5$. Another table gives the values $A_j(r)$ for $0 \leqq j \leqq 5$ and $2 \leqq r \leqq 12$. It is shown that for fixed j and large r, $A_j(r) \sim r!/4^j \cdot j!(r-2j)!$, and an asymptotic formula for $p(n, r)$ is obtained for fixed r and large n.

N. J. Fine (Philadelphia, Pa.).

Romanov, N. P. On asymptotic theorems of the theory of numbers. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **5**, 54–60 (1949). (Russian)

By taking special values in his identity

$$\zeta(s) \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{A_n B_n}{\varphi_s(n)}$$

[Izv. Akad. Nauk SSSR. Ser. Mat. **10**, 3–34 (1946); MR **8**, 9] the author obtains a variety of identities such as

$$\sum_{n=1}^{\infty} \lambda(n, s) x^n = -x\zeta(s) - \sum_{n=1}^{\infty} \frac{\mu(n)}{\varphi_s(n)} \sum_{i=1}^{\varphi(n)} \frac{x}{x - \rho_i^{(n)}},$$

where $\rho_i^{(n)}$ runs through the primitive nth roots of unity, $|x| < 1$, and

$$\lambda(n, s) = \zeta(s) \frac{\varphi_{s-1}(n)}{n^{s-1}} \quad (n > 1); \ \lambda(1, s) = 0.$$

From this he deduces

$$\lim_{x\to\exp(2\pi il/k)} (1-|x|)\sum_{n=1}^{\infty}\lambda(n,s)x^n=\frac{\mu(k)}{\varphi_s(k)},$$

where $(k, l)=1$, and a similar relation involving a function $\Lambda(n, s)$. This is obtained by letting $x\to\exp(2\pi il/k)$ in each term of an infinite series, without anything being said to justify the inversion of limiting operations involved.

Since $\lambda(n, s)\to\Lambda(n)$ as $s\to1$, we should have, if the order of limiting operations $x\to\exp(2\pi il/k)$, $s\to1$, can be inverted, that

$$\lim_{x\to\exp(2\pi il/k)} (1-|x|)\sum_{n=1}^{\infty}\Lambda(n)x^n=\frac{\mu(k)}{\varphi(k)}$$

a result equivalent to the Prime Number Theorem. By using a theorem of Hardy and Littlewood and by means of some rather complicated analysis which is only briefly sketched, it is shown that this inversion can be justified, and that the corresponding result for primes in arithmetic progression can be deduced. It is of interest that while this involves a knowledge of the order of magnitude of Dirichlet L-series and their derivatives and reciprocals it does not require the property $L(1+iy)\neq0$ $(y\neq0)$.

There are numerous misprints. *R. A. Rankin.*

Karamata, J. Evaluation élémentaire des sommes typiques de Riesz de certaines fonctions arithmétiques. Acad. Serbe Sci. Publ. Inst. Math. **7**, 1–40 (1954).

The author obtains estimates of the Rieszian sums

$$\sum_{n\leq x} a_n \log^k\left(\frac{x}{n}\right).$$

In particular, if $na_n=\mu(n)$ and $k=2$, the value of the sum is $2\log x+O(1)$; for the case $na_n=\Lambda(n)$ and $k=3$, he obtains $c_4\log^4 x+\cdots+c_1\log x+O(1)$, where the c's are constants. Both sums are of interest in analytic prime number theory. The proofs are elementary. *S. Chowla.*

Carlitz, L. Some arithmetic properties of the Olivier functions. Math. Ann. **128**, 412–419 (1955).

Let k be a fixed integer $\geqq1$. Olivier [J. Reine Angew. Math. **2**, 243–251 (1827)] defined the functions

$$\text{(1)}\qquad \Phi_r(x)=\sum_{m=0}^{\infty}\frac{x^{mk+r}}{(mk+r)!}\quad(0\leqq r\leqq k-1).$$

The present author defines sets of rational integers $a_m=a_{k,m}$ by means of $1/\Phi_0(x)=\sum_{m=0}^{\infty} a_m x^m/m!$ $(a_0=1)$, so that $a_m=0$ unless $k|m$. He first shows that

$$\text{(2)}\qquad \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}a_{m+sw}\equiv0\pmod{p^{re}, p^m},$$

where $p\nmid k$, $p^{e-1}(p^u-1)|w$, and u is the least exponent such that $p^u\equiv1 \pmod k$. Then he obtains a result like (2) for the coefficients $a_m^{(\lambda)}$ occurring in the power series expansion of $(\Phi_0(x))^{-\lambda}$, where λ is an arbitrary integer. The proof depends upon an explicit formula for $a_m^{(\lambda)}$. Next he proves some general results concerning series that include (1); in particular the results apply to certain combinations of the Jacobi elliptic functions. *A. L. Whiteman.*

Deuring, Max. Zur Transformationstheorie der elliptischen Funktionen. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1954**, 95–104 (1954).

Let $\wp\left(z;\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right)$ denote the Weierstrass elliptic function with periods ω_1, ω_2 and let g_2, g_3 and Δ be the corresponding invariants. The function τ defined by

$$\tau(z;\omega_1,\omega_2)=N\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\wp^{w/2}\left(z;\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right),$$

is basic in the study of the class-fields of imaginary quadratic fields. The factor $N\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$ is a normalizing factor depending on the invariants g_2, g_3 and Δ, and w is the number of units of the field. If P is a 2×2 matrix with integral entries having no factor in common and with determinant p, then a simple argument on zeros and poles yields the identity

$$\tau\left(pz;P\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right)$$
$$=\frac{\tau\left(z;\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right)^p+A_{p-1}^{(P)}\tau\left(z;\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right)^{p-1}+\cdots+A_0^{(P)}}{p^w\tau\left(z;\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right)^{p-1}+B_{p-2}^{(P)}\tau\left(z;\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right)^{p-2}+\cdots+B_0^{(P)}}.$$

The author investigates the nature of the coefficients $A_n^{(P)}$, $B_n^{(P)}$. If $\omega=\omega_1/\omega_2$ and if $j(\omega)=1728g_2^3/\Delta$, then $A_n^{(P)}$ and $B_n^{(P)}$ belong to the field $R(j(\omega), j(P(\omega)))$. In the case where p is a prime, each matrix P can be expressed uniquely in the form $P=MP_\mu$, where M belongs to the full modular group and P_μ is one of the canonical matrices

$$P_0=\begin{pmatrix}p&0\\0&1\end{pmatrix}\quad\text{or}\quad P_\mu=\begin{pmatrix}1&\mu\\0&p\end{pmatrix}\quad(\mu=1,2,\cdots,p).$$

The author proves that the Fourier coefficients of the functions $B_n^{(P)}$ are algebraic integers if $P=P_\mu$, $1\leqq\mu\leqq p$, and rational integers if $P=P_0$. In the special case $P=P_0$, $p=\text{prime}>3$, he also shows that the functions $A_n^{(P)}$ and $B_n^{(P)}$ can be expressed as rational functions of $j(\omega)$ and $j(p\omega)$ using rational coefficients. Furthermore, the Fourier coefficients of $A_n^{(P)}$, $0\leqq n\leqq p-1$, $B_n^{(P)}$, $1\leqq n\leqq p-2$, and $1-B_0^{(P)}$, are rational numbers whose numerators are divisible by p and whose denominators have only the prime factors 2 and 3. A similar result is also obtained for more general P. *T. M. Apostol* (Pasadena, Calif.).

Schoeneberg, Bruno. Über die Quaternionen in der Theorie der elliptischen Modulfunktionen. J. Reine Angew. Math. **193**, 84–93 (1954).

In a previous paper [Math. Ann. **113**, 380–391 (1936)], the author has defined certain theta functions connected with a quaternion algebra, which possess transformation equations of the usual type. In the present work, he considers those theta functions which are modular forms (dimension = 2) of prime level (Stufe). Let

$$\text{(1)}\qquad \vartheta(\tau,\rho,\mathfrak{a},\mathfrak{b})=\sum_{\substack{\mu\equiv\rho\,(\mathfrak{ab})\\ \rho\in\mathfrak{a}}}\exp\left(2\pi i\tau\frac{\mu\mu'}{Aq}\right),\quad \operatorname{Im}\tau>0,$$

where q is a rational prime, $\mathfrak{a}$ is an integral right ideal in $\mathfrak{J}$, a maximal order in the definite (generalized) quaternion algebra $\mathfrak{S}$; $\rho\in\mathfrak{a}$, $\mathfrak{b}$ is the different, $\mathfrak{b}^2=(q)$, $(\mathfrak{a},\mathfrak{b})=1$, $A^2=N(\mathfrak{a})$, and $\mu\mu'$ are not associates. (By a definite quaternion algebra, the author means one in which the product of a number by its conjugate is a definite quadratic form.) The linear family of ϑ obtained by fixing $\mathfrak{a}$ and $\mathfrak{J}$ and varying ρ is partitioned into subsets which are invariant under the substitutions of the irreducible representations of $\mathfrak{M}(q)$, the inhomogeneous modular group mod q. When $\rho=0$,

a linear relation is obtained between the ϑ series and the $\wp$ division values of level q. Two numerical examples are given.

J. Lehner (Los Alamos, N. M.).

Rankin, R. A. Chebyshev polynomials and the modulary group of level p. Math. Scand. 2, 315–326 (1954).

Define

$$T_n(x)=\cosh n\theta,\quad F_n(x)=\sinh n\theta/\sinh\theta,$$

where $x=\cosh\theta$. For p an odd prime let $\bar{G}(p)$ denote the group $\bar{\Gamma}(1)/\bar{\Gamma}(n)$, where $\bar{\Gamma}(1)$ is the full inhomogeneous modular group and $\bar{\Gamma}(n)$ is the inhomogeneous principal congruence group of level n, where n is a positive integer. Put $q=\frac{1}{2}(p-1)$, $r=\frac{1}{2}(p+1)$. Then the order of every element of $\bar{G}(p)$ other than the identity is either p or a divisor of q or r. The subgroups of order p or q are easily described. The main purpose of the present paper is to discuss the subgroups of order r by means of properties of $T_n(x)$ and $F_n(x)$. At the same time a number of congruence properties of these polynomials are obtained. If x is an integer, define the order n_x as the least positive n such that $F_n(x)\equiv 0 \pmod p$. It is proved that for any x, exactly one of the congruences $F_p(x)\equiv 0$, $F_q(x)\equiv 0$, $F_r(x)\equiv 0 \pmod p$ holds; the corresponding classes of residues are denoted by C_p, C_q, C_r, respectively. Then, for example, x is a generator of C_t, $t>1$, if and only if it is of order t and $T_t(x)\equiv -1$; also if x is a generator of C_t, $t>1$, then $T_{t+n}(x)\equiv -T_n(x)$, $F_{t\pm n}(x)\equiv \pm F_n(x)$. It is also proved that if $\xi \in \bar{G}(p)$, $p\neq 2$, $\xi\neq$identity, and S is any matrix representing ξ, then the order of ξ is the same as the order of x, where $2x=\operatorname{tr} S$.

L. Carlitz (Durham, N. C.).

Fryer, K. D. Note on permutations in a finite field. Proc. Amer. Math. Soc. 6, 1–2 (1955).

A recent theorem of Carlitz [same Proc. 4, 538 (1953); MR 15, 3] states that every permutation on the numbers of GF(q) can be derived from the permutation polynomials (1) $\alpha x+\beta$, x^{q-2} ($\alpha,\beta\in$ GF(q), $\alpha\neq 0$). In this paper the author proves the following theorem. The permutations $P: x'=x+1$, $Q: x'=mx^{q-2}$ in GF(q), q prime, generate the symmetric group $\mathfrak{S}_q$ if: (2) m is a square of GF(q), $q=4n+1$, or (3) m is a nonsquare of GF(q), $q=4n+3$, and generate the alternating group $\mathfrak{A}_q$ if: (4) m is a square of GF(q), $q=4n+3$, or (5) m is a nonsquare of GF(q), $q=4n+1$. This result includes the result of Carlitz when q is prime, in that if all α are used in (1), then the author's Q is present.

A. L. Whiteman (Los Angeles, Calif.).

Barbilian, D. Les arithmétiques non commutatives à théorie exhaustive des idéaux. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 4, 257–344 (1953). (Romanian. Russian and French summaries)

The author generalizes Chevalley's number theory in hypercomplex systems [L'arithmétique dans les algèbres de matrices, Hermann, Paris, 1936] by modifying his four axioms in various ways. In addition to the literature quoted, there is relevant material in M. Eichler [J. Reine Angew. Math. 176, 192–202 (1937); Math. Z. 43, 102–109 (1937)].

O. Taussky-Todd (Washington, D. C.).

Have

***Chevalley, C. Class field theory.** Nagoya University, Nagoya, 1954. ii+104 pp.

These are the author's lecture notes for a course given by him at Nagoya University in 1953/54. The purpose of these lectures was to present the new cohomology methods in class-field theory. Indeed, one finds here the first (printed) exposition of class-field theory in which full use of cohomology theory is made.

After a brief discussion of ideles and idele classes, including the requisite topological structure as well as the operations on ideles and idele classes induced by field automorphisms, there follows a complete and self-contained account of abstract cohomology theory for finite groups. The treatment is independent of general homological algebra and uses a formally new definition of the cohomology groups which is skillfully adapted to yield all the important formal properties of the cohomology groups with a minimum of computation.

The significance of the cohomology groups in various modules (multiplicative groups of field elements, ideles, and idele classes) for the class-field theory can already be surmised from the following facts. If G is a finite group and A is a G-module, the 0-dimensional cohomology group $H^0(G,A)$ for G in A is $A^G/N_G(A)$, where A^G is the submodule of A consisting of all the G-fixed elements and N_G is the endomorphism $a\rightarrow\sum_{\sigma\in G}\sigma\cdot a$, when A is written additively (in the applications, N_G is the norm map). The most remarkable general result (due to J. T. Tate) on these cohomology groups which is of particular significance for the class-field theory is the following: let G be a finite group, and let A be a G-module such that, for every subgroup K of G, $H^1(K,A)=(0)$, while $H^2(K,A)$ is cyclic and of the same order as K. Then the cup multiplication with any generator of $H^2(G,A)$ gives an isomorphism of $H^n(G,Z)$ onto $H^{n+2}(G,A)$ for all n, where Z is the additive group of the integers, with trivial G-operators.

The main point of the class-field theory, from this point of view, is that the hypotheses of Tate's theorem are satisfied when G is the Galois group of a finite normal extension L/K of an algebraic number field K, and A is the idele-class group C_L of L. Indeed, the abstract result then yields immediately the reciprocity isomorphism between the norm-class group $C_K/N_{L/K}(C_L)$, which is $H^0(G,C_L)$, and the Galois group G/G' (G the Galois group of L/K, G' its commutator subgroup) of the maximal abelian subextension of L/K, because G/G' is canonically isomorphic with $H^{-2}(G,Z)$. Furthermore, it is possible to pick out a generator of $H^2(G,C_L)$ in a definite canonical fashion, and the various formal properties of the reciprocity map of the class-field theory (for field towers, and for translated extensions) become direct consequences of formal relations between the canonical cohomology generators.

In arithmetic respects, the proofs of these cohomology results involve much the same considerations as those which lead to the fundamental inequalities of the class-field theory, as for instance in the author's paper in Ann. of Math. (2) 41, 394–418 (1940) [MR 2, 38]. In group-theoretical respects, there are here considerable clarifications and simplifications which are due chiefly to the following features of the cohomology approach. The cohomology results hold quite generally for arbitrary normal extensions, so that no artificial restriction to the case of abelian extensions must be made. In fact, the non-abelian features of the theory wash out automatically when the cohomology results are translated into the usual terms. On the other hand, the computation of the various cohomology groups can often be reduced quite easily to the case of cyclic operator groups of prime order by making use of cohomology restriction, transfer and lifting (group→subgroup, subgroup→group, factor group→group, respectively) and of Sylow subgroups and their composition series. Moreover, in the cyclic case,

much power is gained from the dimensional periodicity (of period 2) of the cohomology groups which can be exploited in a far more systematic fashion than is possible with the more direct group-theoretical manipulations that occur in the classical proofs.

The most important device in this connection is the following cohomological form (due to J. T. Tate) of the relevant case of Herbrand's group-theoretical lemma. Let G be a finite cyclic group. A G-module A is called an Herbrand module if $H^0(G, A)$ and $H^1(G, A)$ are finite, and the Herbrand quotient $Q(A)$ of A is then defined as the rational number whose numerator is the order of $H^0(G, A)$ and whose denominator is the order of $H^1(G, A)$. Let A be any G-module and let B be a G-submodule of A. Then, if any two of the modules A, B, A/B are Herbrand modules, so is the third, and $Q(A/B)Q(B)=Q(A)$. The Herbrand quotient of any finite module is 1, whence, if A/B is finite, $Q(A)=Q(B)$. In the case where G is cyclic and of prime order p, and A is finitely generated, the author succeeds in computing $Q(A)$ explicitly in terms of the ranks (as abelian groups) α and β of A and A^G, respectively. The result is: $Q(A)=p^e$, with $e=(p\beta-\alpha)/(p-1)$. These results are used repeatedly (in the place of the more unwieldy Herbrand lemma) in establishing the fundamental inequalities.

In these terms, the essential case of the first fundamental inequality is expressed as follows: let L/K be a cyclic extension of prime degree p, and let C_L denote the idele-class group of L, regarded as a module for the Galois group of L/K. Then the Herbrand quotient $Q(C_L)$ is equal to p. The proof makes use of the above technique of computing Herbrand quotients by considering suitable subgroups of the idele group and the principal-idele group, and the result is deduced from comparatively simple results concerning the local cohomology groups and from the generalized unit theorem.

The second fundamental inequality is then proved for L/K of prime degree p and assuming that K contains the pth roots of unity. As in the classical procedure, the proof of this is the most difficult one. It uses an auxiliary Kummer extension and involves applications of the first inequality, as in the author's old proof.

At this point, the main arithmetical obstacles have been overcome, and the cohomology groups come into the foreground. Using the reduction technique for the computation of cohomology groups that we have mentioned above, it is now easily shown from the preceding results that, for any finite normal extension L/K with Galois group G, one has $H^1(G, C_L)=(0)$.

The main device used in the determination of $H^2(G, C_L)$ consists in first determining this group for the case of a cyclic cyclotomic extension and then to carry over the results to the general case by a comparison technique taking place in a composite of the given extension with a suitable cyclic cyclotomic extension. For cyclotomic extensions, the author begins by explicitly defining the reciprocity map (Artin symbol) for cyclotomic extensions of the rational number field. For arbitrary base fields, one knows a posteriori that the reciprocity map is then determined by the translation property, and it is here proved directly that the translation formula for the Artin symbol indeed yields a meaningful Artin symbol for general cyclotomic extensions which has all the desired properties. In particular, one is then in a position to show that, for a cyclic cyclotomic extension L/K with Galois group G, $H^2(G, C_L)$ is a cyclic group of order $[L:K]$ (because it is isomorphic with $H^0(G, C_L)$), and that a canonical generator for this group is then defined by the requirement that the corresponding homomorphism $G\rightarrow C_K/N_{L/K}(C_L)$ should be the inverse of the reciprocity map. The generalization to arbitrary finite normal extensions is then effected by means of the comparison technique mentioned above whose success depends essentially on the vanishing of the 1-dimensional cohomology groups in the idele-class groups.

Having thus secured the main features of the global class-field theory, the author then defines the local norm-residue symbol from the global Artin symbol so as to secure in advance that the Artin symbol is equal to the product of all the local norm-residue symbols. The fact that the norm-residue symbol has the usual arithmetic significance (i.e. gives the local reciprocity map) is proved by considering the canonical homomorphism of the idele cohomology group into the idele-class cohomology group.

There follows the complete determination of the cohomology groups of dimensions 2 and 3 in the multiplicative groups of the field elements, the ideles, and the idele classes, and of the exact cohomology sequence connecting them. The results contain the classical results on the Brauer algebra classes over algebraic number fields. Finally, the main new result (due to Artin and Tate), a generalization of Hasse's norm theorem, is given: there is a canonical homomorphism of $H^{-3}(G, Z)$ onto the group of the principal ideles of the base field which are local norms modulo the norms of the principal ideles of the extension field. In fact, this homomorphism is obtained by cup multiplication with the canonical image in $H^3(G, L^*)$ of the canonical generator of $H^2(G, C_L)$.

The lectures end with a short proof of the existence theorem, according to which every closed subgroup of finite index in C_K is the group of idele-class norms corresponding to some finite abelian extension of K.

It may be worth while to note the following misprints: p. 58, line 4: instead of ". We have", read: ", and"; p. 70, line 21: the last exponent of the formula should be $2N$; p. 78, second line of third paragraph: replace the second U by J_Q; p. 93, line 10 from bottom: read $N(\mathfrak{p})$ for $N/N(\mathfrak{p})$; p. 103, line 4: read $C_{L'}/N_1$ in the place of B/N_1 (twice).

G. Hochschild (Urbana, Ill.).

Kinohara, Akira. A note on the relative 2-dimensional cohomology group in complete fields with respect to a discrete valuation. J. Sci. Hiroshima Univ. Ser. A. 18, 1–7 (1954).

The main result is an extension of a result of Kawada's [Ann. of Math. (2) **54**, 302–314 (1951), (50) and (51) (localized by (53)), p. 311; MR **13**, 324] from the case of $\mathfrak{p}$-adic number fields to that of a finite separable algebraic extension field K of a field k complete under a discrete valuation, assuming only that the valuation ring of K is generated over that of k by a single element. The cohomology group of the title is computed by means of a normalization procedure relative to such a single generator, while Kawada used two generators (one giving the residue class-field extension, and the other generating the valuation ideal). This results in some simplifications.

G. Hochschild (Urbana, Ill.).

Kinohara, Akira. On the different theorem in complete fields with respect to a discrete valuation. J. Sci. Hiroshima Univ. Ser. A. **18**, 9–12 (1954).

Let K be a finite separable algebraic extension of a field k which is complete under a discrete valuation. The author

gives a proof for the well known result that the different (defined by means of the trace criterion) of K/k coincides with the valuation ring of K if and only if the residue class field extension K'/k' is separable and the ramification index for K/k is 1. His proof involves no novelties and is less transparent and self-contained than that of Artin [Algebraic numbers and algebraic functions, I, Inst. Math. Mech., New York Univ., 1951, Ch. 5, Th. 2; MR 13, 628] to which the author draws attention. *G. Hochschild.*

Rogers, C. A. The Minkowski-Hlawka theorem. Mathematika 1, 111–124 (1954).

Let Λ be a discrete set of points in real Euclidean n-space which does not contain the origin. Where $\delta(\Sigma(r))$ denotes the number of points of Λ within a sphere Σ of radius r divided by the volume of Σ, let $\delta(\Lambda)$ be

$$\lim_{r\to\infty}\sup\,\sup_{\Sigma(r)}\delta(\Sigma(r)).$$

The purpose of this paper is to prove the following result. If S is a set with outer Jordan content $V(S)$ for which $\delta(\Sigma)V(S)<1$. Then a linear transformation α with unit determinant exists so that $\alpha\Lambda$ has no point in common with S. After specializing Λ to be a lattice, the Minkowski-Hlawka theorem follows simply from this result, thus showing that this theorem depends largely on the density properties of the lattice.

Let u be a unit vector and $C(u,\theta)$ be the cone with vertex at the origin whose generators make an angle θ with u. If

$$\delta(C(u,\theta))=\lim_{r\to\infty}\sup\sup_{\Sigma(r)\subset C(u,\theta)}\delta(\Sigma(r)),$$

then $\delta(u)$ is defined as $\lim_{\theta\to 0}\delta(C(u,\theta))$. Most of the paper is concerned with the following theorem which is stronger than the author's main result. If u is a unit vector for which $\delta(u)$, $\delta(-u)$ are finite and $\rho(x)$ is a function, integrable in the Riemann sense over the whole space, then, for $\epsilon>0$, a linear transformation α exists with unit determinant for which

$$\sum_{x\in\Lambda}\rho(\alpha x)<\delta(u)\int_P\rho(x)\,dx+\delta(-u)\int_N\rho(x)\,dx+\epsilon,$$

where P and N are the half-spaces $x_n=0$, $x<0$.

The proof makes use of transformations

$$\alpha x=(\omega x_1+\omega\Phi_1x_n,\cdots,\omega x_{n-1}+\omega\Phi_{n-1}x_n,\omega^{-n+1}x_n).$$

$I(x)$ is defined as

$$\int_{-\Phi}^{+\Phi}\cdots\int_{\Phi}^{+\Phi}\rho(\alpha x)\,d\Phi_1\cdots d\Phi_{n-1}.$$

By a series of elementary estimates which exploit the density properties of Λ, it is shown that

$$\sum_{x\in\Lambda}I(x)\leqq(2\Phi)^{n-1}\left[\delta(u)\int_P\rho(x)\,dx+\delta(-u)\int_N\rho(x)\,dx+\epsilon\right].$$

As

$$\sum_{x\in\Lambda}I(x)=\int_{-\Phi}^{+\Phi}\cdots\int_{-\Phi}^{+\Phi}\sum_{x\in\Lambda}\rho(\alpha x)\,d\Phi_1\cdots d\Phi_{n-1}$$

it follows that at least one set of values $\Phi_1,\cdots,\Phi_{n-1}$ exist which define an appropriate α. Proofs are indicated of generalizations of this result in which $\rho(x)$ is assumed to be integrable in the Lebesgue sense. A generalization of the Minkowski-Hlawka theorem is given in which the hypothesis involves the $\delta(u)$ concept. *D. Derry.*

Hlawka, Edmund. Zur Theorie der Überdeckung durch konvexe Körper. Monatsh. Math. 58, 287–291 (1954).

Four estimates are obtained for the inhomogeneous minimum M_I of a convex body K in n-dimensional space relative to a lattice G. If K is given by $f(\mathbf{x})\leqq 1$, where f is homogeneous of degree 1, then M_I is defined by

$$M_I=\max_{\mathbf{p}}\min_{\mathbf{g}}f(\mathbf{p}+\mathbf{g}),$$

where $\mathbf{p}$ is any point and $\mathbf{g}$ any point of G. One of the estimates is expressed in terms of the number B of pairs of points of G, other than O, in the interior of K; it is

$$2M_I\leqq w^{1/n}\{w\}^{1-1/n},\quad w=2^n(B+1)\Delta V^{-1},$$

where $\{w\}$ denotes the least integer $\geqq w$ and $\Delta=\det G$, $V=$volume of K. Another estimate, which follows from this, is

$$2M_I\leqq v^{1/n}\{v\}^{1-1/n}\mu_1,\quad v=2^n\Delta V^{-1}\mu_1^{-n},$$

where $\mu_1=\min f(\mathbf{g})$ over $\mathbf{g}\neq O$. All the estimates are proved by applying Minkowski's fundamental theorem, or a similar theorem, to a suitable cylindrical body in $n+1$ dimensions, constructed in terms of K and an arbitrary point $\mathbf{p}$. *H. Davenport* (London).

Watson, G. L. The representation of integers by positive ternary quadratic forms. Mathematika 1, 104–110 (1954).

Let f be a positive definite ternary quadratic form of determinant d and whose coefficients are integers with g.c.d. equal to 1. The author defines n to be an exceptional integer if $f\equiv n \pmod m$ is solvable for all integers m and $f=n$ is not solvable in integers. An exceptional integer is called primitively exceptional if it is not of the form $n_1^2n_2$ with $n_1>1$ and n_2 exceptional. Let $E(f)$ and $E_0(f)$ be the number of exceptional and primitively exceptional, respectively, integers of f. The author proves: for any positive δ and sufficiently large d, $E_0(f)>d^{1-\delta}$. *B. W. Jones.*

O'Meara, O. T. Quadratic forms over local fields. Amer. J. Math. 77, 87–116 (1955).

The author is concerned with integral equivalence of quadratic forms over local fields F, that is, fields which are complete under a discrete non-Archimedean valuation and have a perfect residue class field. He defines $e=\text{ord }(2)$ to be the ramification index of F and denotes by $\mathfrak{o}$ the ring of integers in F. A lattice L is said to be integral if $N(L)\subseteq\mathfrak{o}$ and totally integral if an addition $X\cdot Y\in\mathfrak{o}$ for all X and Y in L. He proves the following cancellation theorem: If $e\leqq 1$ and L is a totally integral lattice having the two decompositions $L=L_0\oplus L_1=K_0\oplus K_1$ in which L_0 and K_0 are isometric and of unit determinant, then L_1 and K_1 are isometric provided that their norms are equal.

A complete set of invariants is found for L in the unramified case with 2 a prime in F. In the ramified case he proves that if L and K are totally integral lattices of unit determinant in the same metric space V, then L is isometric to K provided that L and K represent the same numbers. The problem remains of finding a complete set of invariants for forms of arbitrary determinant over a general local field. *B. W. Jones* (Boulder, Colo.).

Žmud', È. M. On integer transformations of quadratic forms. Mat. Sb. N.S. 32(74), 287–344 (1953). (Russian)

The author says that the integral quadratic form f in n variables is m-derived from the integral form g if it arises

from g by an integral transformation of determinant m. If f is not the m-derivative of any form it is m-simple. For prime m the author determines all the genera which can be m-derived from a given genus, and also gives a criterion for m-simplicity. Finally for m prime he gives a formula of the type $h(\mathfrak{G}')M(\mathfrak{G}')=\sum j(\mathfrak{G},\mathfrak{G}')M(\mathfrak{G})$ for the mass of a genus $\mathfrak{G}'$ in terms of the masses of the genera $\mathfrak{G}$ from which it is m-derivable. Here $h(\mathfrak{G}')$ and $j(\mathfrak{G},\mathfrak{G}')$ are integers which are given explicitly in terms of the invariants of $\mathfrak{G}'$ and $\mathfrak{G}$, $\mathfrak{G}'$ respectively. The work depends heavily on the results and techniques of Minkowski's Paris Prize Essay [Mém. Acad. Sci. Inst. France (2) **29**, no. 2 (1884)], which is the most modern reference cited (apart from the author's own dissertation and an elementary text). The author is apparently unaware of more recent relevant work: the series of papers by Meyer [J. Reine Angew. Math. **98**, 177–230 (1885); **103**, 98–117 (1888); **108**, 125–139 (1891); **112**, 87–88 (1893); **113**, 186–206 (1894); **114**, 233–254; **115**, 150–182 (1895); **116**, 307–325 (1896)]; the work of H. Brandt and his school [see, e.g., Brandt, Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, Füssli, Zürich, 1945, pp. 87–104; MR **8**, 138] and of Eichler [Quadratische Formen und orthogonale Gruppen, Springer, Berlin, 1952; MR **14**, 540]. *J. W. S. Cassels* (Cambridge, England).

van der Blij, F. Even quadratic forms with determinant unity. Quart. J. Math., Oxford Ser. (2) **5**, 297–300 (1954).

Let $q=e^{\pi i\tau}$, $|q|<1$, set $\vartheta_{ij}(0|\tau)=\vartheta_{ij}$ $(i,j=0,1)$ and define

$$A_k=\tfrac{1}{2}(\vartheta_{00}^{8k}+\vartheta_{01}^{8k}+\vartheta_{10}^{8k}),\quad B_k=\sum_{(m_i)} q^{Q(m_1,\cdots,m_{8k})},$$

where $Q(m_1,\cdots,m_{8k})$ is a quadratic form in $8k$ variables, with even coefficients and determinant unity. The following three theorems are proven: (1) $A_1{}^2=A_2$. (2) There exist forms $Q_k{}^*$ such that the corresponding function $B_k{}^*$ equals A_k. (3) The $[k/3]+1$ functions $A_1{}^{k-3\alpha}A_3{}^{\alpha}$ with fixed integer k are linearly independent and A_k can be represented as a sum of these functions, with constant coefficients. (1) follows from the well-known relation $\vartheta_{00}{}^4=\vartheta_{01}{}^4+\vartheta_{10}{}^4$, for which the author gives a simple proof. (2) is proven by actually constructing Q^*, and (3) by induction on k. Some further relations are established between the A_k's, the Eisenstein series of dimension -12 and $\Delta_3=q\prod_{n=1}^{\infty}(1-q^n)^{24}$.

E. Grosswald (Philadelphia, Pa.).

ANALYSIS

Aljančić, S., Bojanić, R., et Tomić, M. Sur la valeur asymptotique d'une classe des intégrales définies. Acad. Serbe Sci. Publ. Inst. Math. **7**, 81–94 (1954).

An investigation is made of the asymptotic behaviour of

$$\Phi(\lambda)=\int_a^b f(t)L(\lambda t)\,dt,\quad 0\leq a<b\leq\infty,$$

as $\lambda\to+\infty$. A large number of results are proved of which the following are typical: (i) if $L(t)$ is the product of two monotone functions of slow growth and if $\int_a^\infty t^\eta|f(t)|\,dt$ exists for some $\eta>0$, then $\Phi(\lambda)\sim L(\lambda)\int_a^\infty f(t)\,dt$; (ii) if $L(t)$ is of slow growth, monotone, convex, and $L(+\infty)=0$, and if $\int_0^x(1-tx^{-1})f(t)\,dt\sim A$ as $x\to+\infty$, then $\Phi(\lambda)\sim L(\lambda)A$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

McMillan, Brockway. Absolutely monotone functions. Ann. of Math. (2) **60**, 467–501 (1954).

The author generalizes the theorem of S. Bernstein: if $F(x)$ is a real-valued function with all its differences non-negative on $0\leq x<1$, then

$$(*)\qquad F(x)=F(0)+\sum_{n=1}^{\infty}\frac{1}{n!}x^nF^n(0).$$

He permits $F(x)$ to be valued in a partially ordered linear space L, assumed to be conditionally complete (in its positive cone) and he allows x to vary over an abstract set S of objects, for which a suitable class of partitions is postulated. The expression $x^nF^n(0)$ in (*) is redefined in terms of limits of nth order differences, using the facts that F has all differences non-negative and the positive cone in L is conditionally complete. Then (*) is established but with $\geq$ in place of $=$; for the special case that x varies over a real interval, even $=$ holds, but examples are given of more general S for which $=$ fails to hold. *I. Halperin.*

Ionescu, D. V. Formules de cubature, le domaine d'intégration étant un triangle quelconque. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 423–430 (1953). (Romanian. Russian and French summaries)

The author investigates numerical integration formulas of the type

$$\iint_T\phi(x,y)\,dxdy=\sum_{i=1}^{n}P_i[\phi(L_i)+\phi(M_i)+\phi(N_i)],$$

where T is a triangle; L_i, M_i, N_i are the centers of mass of the respective systems of masses $(\alpha_i,1,1)$, $(1,\alpha_i,1)$, $(1,1,\alpha_i)$ placed at the vertices of T; P_i are constants; and the formula is to be exact for all polynomials of degree at most n. The problem of finding the P_i is impossible if $n>5$. For $n\leq 5$ it is possible, except for certain exceptional cases, when all the α_i are different; there are no exceptions if one $\alpha_i=1$.

R. P. Boas, Jr. (Evanston, Ill.).

Sierpiński, W. Sur une relation entre deux substitutions linéaires. Fund. Math. **41**, 1–5 (1954).

J. von Neumann has shown that there are linear fractional transformations ϕ_1, ϕ_2 which satisfy no condition of the form

$$\phi_1{}^{k_1}\phi_2{}^{k_2}\phi_1{}^{k_3}\phi_2{}^{k_4}\cdots\phi_1{}^{k_{2n-1}}\phi_2{}^{k_{2n}}=1,$$

where the k_i's are integers, and are $\neq 0$ except perhaps for k_1 and k_{2n} [Fund. Math. **13**, 73–116 (1929)]. (Here the indicated "multiplication" is composition of functions, and 1 denotes the identity.)

The author shows the following. (1) If $\phi_i(x)=a_ix+b_i$ $(i=1,2)$, where $a_i\neq 0$, then

$$\psi=\phi_1\phi_2{}^2\phi_1{}^{-1}\phi_2{}^{-1}\phi_1\phi_2{}^{-1}\phi_1{}^{-1}\phi_2{}^2\phi_1\phi_2{}^{-1}\phi_1{}^{-1}\phi_2{}^{-1}=1.$$

(2) There is no system of $2q-1\leq 11$ integers $k_1, k_2, \cdots, k_{2q-1}$, different from 0 except perhaps for k_{2q-1}, such that for every $\phi_i(x)=a_ix+b_i$ $(i=1,2;\ a_i\neq 0)$, the function

$$\phi_1{}^{k_{2q-1}}\phi_2{}^{k_{2q-2}}\cdots\phi_2{}^{k_2}\phi_1{}^{k_1}$$

is the identity. *E. E. Moise* (Ann Arbor, Mich.).

Theory of Sets, Theory of Functions of Real Variables

Erdös, Paul. Some remarks on set theory. IV. Michigan Math. J. **2** (1953–54), 169–173 (1955).

[For part III see same J. **2**, 51–57 (1954); MR **16**, 20.] Some of the results contained in this paper are the following. (1) Assume that $2^{\aleph_0}=\aleph_1$. Decompose the set of all lines in the plane into two arbitrary mutually exclusive sets L_1 and L_2. Then there exists a decomposition of the plane into two sets S_1 and S_2 such that each line belonging to L_i intersects S_i ($i=1, 2$) in an at most enumerable set. This generalizes a result due to Sierpiński [Hypothèse du continu, Warszawa-Lwów, 1934, p. 9]. (2) If A and B are sets of real numbers, let $A\oplus B$ denote the set of all numbers $a+b$ with $a \in A$ and $b \in B$. Assume that $2^{\aleph_0}=\aleph_1$. Suppose that $\{S_\xi\}$, $\xi<\omega_1$, is a family of sets of real numbers such that the set of all real numbers is not the union of enumerably many sets $S_{\xi_k}\oplus\{a_k\}$ ($\xi_k<\omega_1$, a_k real, $k=1, 2, \cdots$). Then there exists a set A of power $\aleph_1$ such that none of the sets $S_\xi\oplus A$ ($\xi<\omega_1$) is the set of all real numbers. (3) Let M be a set, μ be a measure defined on some subsets of M, and M be the union of enumerably many sets of finite measure. Then the family of subsets of M of positive measure has the property that to each nonenumerable collection of sets in this family there corresponds a nonenumerable subcollection such that each pair of sets in this subcollection has a nonempty intersection. This result is related to a problem posed by Marczewski [Fund. Math. **34**, 127–143 (1947), pp. 129–130; MR **9**, 98]. *F. Bagemihl* (Princeton, N. J.).

Newns, W. F. A theorem on cardinal numbers. Edinburgh Math. Notes no. **39**, 4–5 (1954).

Given a set A, let $a \in A$ and

$$T=\{S: S\subseteq A, |S|\geqq|A-\{a\}|\}.$$

The purpose of this note is to prove, without the use of the axiom of choice, that $|T|>|A|$. *F. Bagemihl.*

Ginsburg, Seymour. Uniqueness in the left division of order types. Proc. Amer. Math. Soc. **6**, 120–123 (1955).

As sequel to some results by Sierpiński [Fund. Math. **35**, 1–12 (1948); MR **10**, 358] and his own [Trans. Amer. Math. Soc. **74**, 514–535 (1953); Proc. Amer. Math. Soc. **5**, 554–565 (1954); MR **14**, 853; **16**, 21] the author establishes the following theorems. Let M be any chain with a fixed-point property. If α, β are order types so that $\alpha<\alpha\cdot 2$, $\alpha\cdot tM=\beta\cdot tM$, then $\alpha=\beta$ (Th. 1). The hypothesis on α is needed as is evident from the case (order type of M) $tM=1+\omega^*+\omega$, $\alpha=\eta$, $\beta=\eta+1$. But if, moreover, M has a first element as well as a last element, then the statement of the theorem 1 holds for any α (Th. 2). *G. Kurepa.*

Smith, Newton B. Types of functions. Proc. Iowa Acad. Sci. **61**, 324–329 (1954).

The real function $f(x)$ defined on the real interval X is said to be neighborly [cliquish] at the point $\xi \in X$ if, given $\epsilon>0$ and the open interval $N(\xi)\supset\xi$, there exists an open interval $N\subset N(\xi)\cdot X$ such that

$$|f(x)-f(\xi)|<\epsilon \quad [|f(x')-f(x)|<\epsilon]$$

for $x, x' \in N\cdot X$. f is discriminative at ξ if, given $\epsilon>0$ and $N(\xi)$, there exist an η and an $N\subset N(\xi)\cdot X$ such that $|\eta-f(\xi)|<\epsilon$ and $f(x)\neq\eta$ for $x \in N$. Theorem 1: If f is bounded, discriminative and has the Darboux property on X, then f is cliquish on X. Theorem 3: If f is of Baire class less than two and has the D-property on the open interval I, then f is neighborly on I. *M. Cotlar* (Mendoza).

Aquaro, Giovanni. Sul criterio di Arzelà per la continuità del limite di una successione convergente di funzioni continue. Rend. Sem. Fac. Sci. Univ. Cagliari **24**, 10–13 (1954).

A sequence f_n of functions on a set S to a metric space M is said to be "quasi-uniformly convergent" to the function f provided f_n converges pointwise to f, and for each positive ϵ and positive integer v there is a bounded integer-valued function v_ϵ on S such that for each $x \in S$, $v_\epsilon(x)\geqq v$ and $|f(x)-f_{v_\epsilon(x)}(x)|<\epsilon$. Suppose there is associated with S a function $'$ on 2^S to 2^S such that $(E-\{a\})'=E'\subset F'$ whenever $a \in E\subset F\subset S$, so that S is a (V)-space in the sense of Fréchet [Les espaces abstraits, Gauthier-Villars, Paris, 1928, pp. 172–181]. Fréchet observed [loc. cit., p. 247] that if S is self-compact [loc. cit., p. 231], each f_n is continuous, f_n converges pointwise to f, and the derivation $'$ satisfies certain additional conditions which make S into an (H)-space [loc. cit., p. 185], then f is continuous if and only if the convergence is quasi-uniform. He asked [loc. cit., p. 247] whether the additional conditions could be avoided. The author gives a partial answer, by proving the theorem for a (V)-space in which $'$ is distributive; i.e., $(E\cup F)'=E'\cup F'$ for all $E, F\subset S$. *V. L. Klee, Jr.* (Seattle, Wash.).

Gładysz, S. A random ergodic theorem. Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 411–413 (1955).

The purpose of this note is to state (without proof) several random ergodic theorems. The first of these is typical, and, presumably, the others are relatively easy consequences of it; it runs as follows. Let T be a measurable and measure-preserving transformation of a measure space X (of total measure 1) into itself; suppose that for each x in X there is defined a measurable and measure-preserving transformation $s\rightarrow sx$ of a measure space S (of total measure 1) into itself. If $(s, x)\rightarrow sx$ is a measurable function of its two arguments and if f is an integrable function on $S\times X$, then

$$F(s, x)=\lim_n \frac{1}{n}\sum_{k=0}^{n} f(s\cdot x\cdot Tx\cdots T^{k-1}x, T^kx)$$

exists for almost all (s, x), and F is integrable on $S\times X$. *P. R. Halmos* (Chicago, Ill.).

Moore, Marian A. Approximations of ϕ-integrals by Riemann and Darboux sums, and other contributions to the theory of ϕ-integrals in general spaces. Riv. Mat. Univ. Parma **5**, 99–123 (1954).

The paper is, in effect, an addendum to the book of Hahn and Rosenthal [Set functions, Univ. of New Mexico Press, Albuquerque, N. M., 1948; MR **9**, 504]. Using the notation and the terminology of that book, the author extends (from the finite to the σ-finite case) some mean-value theorems, some convergence theorems, and several results concerning the approximation of integrals by sums. *P. R. Halmos.*

Koksma, J. F. Estimations de fonctions à l'aide d'intégrales de Lebesgue. Bull. Soc. Math. Belg. **6** (1953), 4–13 (1954).

The following general theorem is proved: Let $\{u_\nu(x)\}$ be a sequence of real functions in $L^{(p)}(a, b)$, $p>1$. Let $\eta(t)$ be a positive decreasing function of $t>0$, such that $\sum_{N=1}^{\infty}\eta(N)/N<\infty$. Let C_1, C_2, C_3, α, β, γ be non-negative constants with $\alpha>0$, $\gamma>1$. Put $F(M, N)=\sum_{M+1}^{M+N} u_\nu(x)$, and suppose that for every $M\geqq 1$, $N\geqq 1$ such that $M=2^n-1$

and $N|(M+1)$, the inequality

$$\int_a^b |F(M, N)|^p dx \leqq C_1 N^\alpha G(M, N) + C_2 (M+N)^{p-\gamma} N^\gamma \eta(N)$$

holds, where G is such that

$$\sum_{k=0}^{(M+1)/N} G(M+kN, N) \leqq C_3 N^\beta G(M, M+1)$$

and $\sum_1^\infty G(2^n-1, 2^n)/2^n(p-\alpha-\beta) < \infty$. Then for almost all $x \in [a, b]$, $F(0, N) = o(N)$.

As an application, it is shown that if $g \in L^{(2)}(0, 1)$ is of period 1 and has Fourier coefficients c_k such that $\sum_{k=1}^\infty |c_k|^2 \sum_{d|k} d^{-1} < \infty$, then

$$N^{-1} \sum_{n=1}^N g(nx) \to \int_0^1 g(t)\, dt$$

for almost all x. This improves an earlier result of the author [J. Indian Math. Soc. (N.S.) **15**, 87–96 (1952); MR **13**, 827].
W. J. LeVeque (Ann Arbor, Mich.).

Federer, Herbert. On Lebesgue area. Ann. of Math. (2) **61**, 289–353 (1955).

Given a Fréchet surface S in terms of a representation $T: Q \to E_3$, where Q is the unit square in the uv-plane and E_3 is Euclidean 3-space, denote by T_x, T_y, T_z the product of T with the orthogonal projections upon the yz, zx, xy planes respectively. We have then the fundamental Cesari inequality $L(T) \leqq L(T_x) + L(T_y) + L(T_z)$, where L stands for Lebesgue area [for definitions and references, see the reviewer's book, Length and area, Amer. Math. Soc. Colloq. Publ. vol. 30, New York, 1948; MR **9**, 505]. The main purpose of the present paper is to establish the validity of the Cesari inequality for Fréchet surfaces given by representations of the form $T: R \to E_n$, where R is any finitely triangulable subset of the plane and n is an arbitrary integer $\geqq 2$. This extension requires the use of novel and involved methods. Many of the partial results derived in the paper apply to Fréchet varieties of arbitrary dimension located in Euclidean n-space E_n.
T. Radó.

Theory of Functions of Complex Variables

Hasimoto, Keizo, Kato, Sadao, and Matuura, Syozo. The second principal theorem in the continuous functions. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **4**, 324–331 (1954).

Let $w(z) = u(x, y) + iv(x, y)$ $(z = x + iy)$ be a one-valued continuous function in a schlicht domain of finite connectivity, except for at most a finite number of poles. The authors obtain five theorems for such continuous mappings of which a typical one is that the sum of the orders of the branch points amounts to twice the valency mean.
M. S. Robertson (New Brunswick, N. J.).

Arsove, Maynard G. On the definition of an analytic function. Amer. Math. Monthly **62**, 22–25 (1955).

Eliminating as much of the real variable theory as possible, the author proves the following form of the Looman-Menchoff theorem: If $f = u + iv$ is continuous on an open set Ω, if $\limsup_{\zeta \to z} |f(\zeta) - f(z)| \cdot |\zeta - z|^{-1}$ is finite for all points $z \in \Omega$ except possibly for a countable set in Ω, and if the Cauchy-Riemann equations hold almost everywhere in Ω, then f is analytic on Ω.
A. J. Lohwater.

Denjoy, Arnaud. Le théorème de Cauchy-Goursat. C. R. Acad. Sci. Paris **240**, 386–389 (1955).

A proof of the Cauchy-Goursat theorem in a very general form in which the topological and metric parts of the proof are kept separate.
A. J. Lohwater (Helsinki).

Denjoy, Arnaud. Sur l'intégrale de Cauchy. C. R. Acad. Sci. Paris **240**, 473–476 (1955).

The Cauchy integral formula is derived under the very general conditions laid down in the paper of the preceding review.
A. J. Lohwater (Helsinki).

Pogorzelski, W. Problème aux limites d'Hilbert généralisé. Bull. Acad. Polon. Sci. Cl. III. **2**, 367–370 (1954).

In the complex plane consider Jordan curves $L_1, \cdots, L_p$, disjoint and bounding disjoint domains $S_1, \cdots, S_p$, respectively. Let L_0 be a Jordan curve enclosing $L_1, \cdots, L_p$; S_0^- is the infinite domain exterior L_0; S^+ is the finite domain bounded by $L_0, L_1, \cdots, L_p$. The problem studied is the following: to find functions $\Phi_1(z), \cdots, \Phi_m(z)$, each analytic in $S^+, S_0^-, S_1^-, \cdots, S_p^-$ separately, such that on $L = L_0 + \cdots + L_p$ one has

$$\Phi_\alpha^+ = G_\alpha \Phi_\alpha^- + \lambda F_\alpha[t, \Phi_1^+, \cdots, \Phi_m^+, \Phi_1^-, \cdots, \Phi_m^-] \quad (\alpha = 1, \cdots, m),$$

the $G_\alpha(t)$ ($\neq 0$ on L) being in H (Hölder class), the $F_\alpha(t, u_1, \cdots, u_{2m})$ being in H in t (on L) and of class Lipschitz in the u_j ($|u_j| \leqq R$); the L_i $(i = 0, \cdots, p)$ have continuous tangents. The method indicated is based on the use of integral equations in the sense of principal values, together with J. Schauder's fixed-point theorem. A detailed development is to appear later.
W. J. Trjitzinsky.

Šmul'yan, Yu. L. Riemann's problem with Hermitian matrices. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 243–248 (1954). (Russian)

Let $A(t)$ be a hermitian matrix of n^2 elements which are functions of Hölder class on $|t| = 1$, while $\det A(t) \neq 0$. Consider the Riemann problems: (1) $\phi^+ = A\phi^-$, (2) $\psi^+ = A^*\psi^-$. Let $X(z)$ be the canonical matrix for (1) and $h_1 \geqq \cdots \geqq h_n$ the indices of problem (1); $\Lambda(z)$ is the matrix $(\delta_{ij} z^{h_i})$. Let $Y(z) = X^{*-1}(\bar{z}^{-1})\Lambda(z)$ be the canonical matrix for (2), the $-h_j$ being the indices for (2). If A is hermitian, $h_1 = -h_n$, $h_2 = -h_{n-1}, \cdots$. If ϕ, ψ are solutions of (1), (2) of orders at infinity h', h'', then: (a) if $h' + h'' = 0$, we have $[\varphi, \psi] = C_1 z^{-h'}$; (b) if $h' + h'' > 0$, we have $[\varphi, \psi] = 0$. Here

$$[\varphi, \psi] = (\phi(z), \psi(\bar{z}^{-1}),$$

the latter symbol $(\cdots, \cdots)$ denoting the scalar product of two vectors. If the quadratic form of the non-degenerate matrix A has in its canonical form p positive and q negative squares, A is of type (p, q); the simplest such matrix is $J_{p,q} = \begin{pmatrix} I_p, & 0 \\ 0, & -I_q \end{pmatrix}$. If $A(t)$ is of type (p, q) and (1) has n_+ positive indices, then $n_+ \leqq \min\{p, q\}$. If $A(t)$ is of type (p, q), problem D is: to find $F(z)$, analytic for $|z| < 1$, $\det F(z) \neq 0$, so that $FJ_{p,q}F^* = A$ on $|t| = 1$. In order that problem D have a solution it is necessary and sufficient that all the indices of the Riemann problem, with matrix $A(t)$, be zero; if so, then a solution of problem D is known, except for an arbitrary constant right multiplier U for which $UJ_{p,q}U^* = J_{p,q}$.
W. J. Trjitzinsky (Urbana, Ill.).

Noble, M. E. A further note on Taylor series with gaps. J. London Math. Soc. **30**, 220–228 (1955).

Notation: $f(z) = \sum a_n z^n$ in $|z| < 1$; $\{n_k\}$ and $\{N_k\}$ are increasing sequences of integers, with $N_k - n_k \to \infty$ and with

$a_m=0$ for $n_k<m\leqq N_k$ $(k=1, 2, \cdots)$; $\phi(n)$ is the least concave majorant of $\log(|a_n|+2)$; $S_k(z)=\sum_{n=0}^{n_k} a_n z^n$. The author continues his search for conditions on f which imply convergence of the sequence S_k at points on $|z|=1$. His main result makes fairly heavy demands on f, but does not require very long gaps: There exist positive numbers $A=A(\delta)$ and $c=c(\delta)<1$ such that if

$$\liminf (N_k-n_k)/\phi(N_k)\geqq A(\delta)$$

then the relation $|S_k(z)-f(z)|=O(c^{N_k-n_k})$ holds uniformly on every closed arc of regularity of f on $|z|=1$. The weaker condition that f be continuous in some closed disc tangent from the interior to the unit circle at $z=1$ implies that $S_k(1)\to f(1)$, provided $N_k>(1+\lambda)n_k$ $(k=1, 2, \cdots)$ for some positive λ. If f is regular in some region

$$|z-1|<\delta, \quad |\arg(z-1)|>\theta$$

(where $\theta<\pi/2$) and continuous in the closure of this region, and if $\phi(N_k)=o(N_k-n_k)$, then $S_k(1)\to f(1)$. The last theorem extends a recent result of Gaier [Trans. Amer. Math. Soc. 75, 48–68 (1953); see Theorem 1; MR 15, 113].

G. Piranian (Ann Arbor, Mich.).

Jenkins, James A. On Bieberbach-Eilenberg functions. II. Trans. Amer. Math. Soc. 78, 510–515 (1955).

[For part I see same Trans. 76, 389–396 (1954); MR 16, 24.] Let C be the class of functions $f(z)=a_1z+a_2z^2+\cdots$, regular in $|z|<1$ and such that $f(z_1)f(z_2)\neq 1$ for $|z_1|<1$, $|z_2|<1$. For $0<\lambda<1$ let $C(\lambda)$ be the subclass of C, for which $|a_1|=\lambda$. Let $M(f)=\sup_{|z|<1}|f(z)|$. It was proved by Rogosinski [J. London Math. Soc. 14, 4–11 (1939)] that $f(z)$ is subordinate to a function $\bar{f}(z)=\bar{a}_1z+\bar{a}_2z^2+\cdots \in C$, for which $|\bar{a}_1|\geqq|a_1|=\lambda$. Hence if $\mu(\bar{f})$ is the lower bound of moduli of values w not taken by $w=\bar{f}(z)$, then $\mu(\bar{f})\geqq\frac{1}{4}|\bar{a}_1|\geqq\frac{1}{4}\lambda$. Thus $M(f)\leqq M(\bar{f})\leqq 1/\mu(\bar{f})\leqq 4/\lambda$.

The author now obtains the exact lower bound $\mu(\lambda)$ for $\mu(\bar{f})$, when $\bar{f}(z)\in C(\lambda)$ and is univalent, showing that $\mu(\lambda)$ increases with λ. The extremal maps are characterized in terms of certain quadratic differentials. The resulting inequality $M(f)\leqq 1/\mu(\lambda)$ for all $f(z)\in C(\lambda)$ is sharp, but equality is never attained.

W. K. Hayman.

Hervé, Michel. Sur les valeurs omises par une fonction méromorphe. C. R. Acad. Sci. Paris 240, 718–720 (1955).

The following notations are used in the paper: $f(z)$ is a meromorphic function in a plane domain D, and E a closed point set of capacity zero lying on the boundary C of D. For $\zeta\in C$, $\Delta(\zeta)$ denotes the cluster set and $R(\zeta)$ the range of values of $f(z)$ at the point ζ. For $\zeta_0\in E$, $\Gamma_E(\zeta_0)$ designates the accumulation set of $\Delta(\zeta)$ as ζ tends to ζ_0, $\zeta\in C-E$.

Kametani [Proc. Imp. Acad. Tokyo 17, 429–433 (1941); MR 7, 380] has proved that the exceptional values at ζ_0 (i.e., values not belonging to $R(\zeta_0)$) which lie in a connected part of $(\Delta-\Gamma_E)(\zeta_0)$ form a set of inner capacity zero. In the special case that D is simply-connected, Noshiro [J. Math. Soc. Japan 1, 275–281 (1950); MR 13, 224] has sharpened this result by proving that the said set cannot contain more than two values. In the present paper the author extends this result of Noshiro. It is shown that the above set contains at most two values also under the more general condition that every point of E belongs to a continuum disjoint to D.

O. Lehto (Helsinki).

Clunie, J. On the determination of an integral function from its Taylor series. J. London Math. Soc. 30, 32–42 (1955).

The author studies the behavior of an entire function f and its derivatives in the neighborhood of points of maximum modulus of f on circles $|z|=r$. Let $M=M(r)$ denote the maximum of $|f(z)|$ on $|z|=r$, and $N=N(r)$ the index of the maximum term in the power series $\sum a_nz^n$ for $f(z)$ on $|z|=r$. An estimate for the difference between $f(z)$ and suitable sums $\sum_{N-k}^{N+k} a_nz^n$ is used to prove the following principal result. One has

$$f^{(q)}(ze^{\tau})=(N/ze^{\tau})^q e^{N\tau}f(z)\{1+\omega(\tau)\},$$

with $|\omega(\tau)|<K/\psi(N)$, whenever $|f(z)|\geqq MN^{-\beta}$ $(\beta\geqq 0)$, $|z|$ non-exceptional in a certain sense, and

$$|\tau|<\frac{K}{\psi(N)N^{1/2}\log^{5/2}N}, \quad q<\frac{KN^{1/2}}{\log^{3/2}N} \quad (N>K).$$

Here ψ is an arbitrary function satisfying

$$K>1/\psi(N)>KN^{-\gamma}\log^{1/2}N$$

for some $\gamma>0$. The proofs are similar to those given previously by the author for the case $q=0$, $f(z)$ of finite order [same J. 28, 58–66 (1953); MR 14, 547].

J. Korevaar.

Fuchs, W. H. J. On the growth of functions of mean type. Proc. Edinburgh Math. Soc. (2) 9, 53–70 (1954).

The author gives a complete solution to a problem of V. Bernstein concerning the growth of functions of exponential type in a half-plane. Let $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_{n+1}-\lambda_n\geqq C>0$ for $n=1, 2, \cdots$, with distribution function $\Lambda(r)=$ number of $\lambda_n\leqq r$. Let $l(r)=\Lambda(r)-r$ and set

$$\phi(r)=\int_1^r u^{-1}dl(u)=r^{-1}l(r)+\int_1^r u^{-2}l(u)\,du+O(1).$$

Let f be any function which is regular in the right half-plane, continuous in its closure, not identically zero, and such that

$$|f(z)|<\exp(B|z|), \quad |f(iy)|<\exp(\pi|y|).$$

Conditions on $l(r)$ are known which insure that

$$\limsup x^{-1}\log|f(x)|=\limsup(\lambda_n)^{-1}\log|f(\lambda_n)|$$

[Levinson, Gap and density theorems, Amer. Math. Soc. Colloq. Publ., v. 26, New York, 1940; MR 2, 180; R. P. Boas, Duke Math. J. 13, 471–481 (1946); MR 8, 372].

The author strengthens these by showing that the following two conditions are necessary and sufficient for the property to hold: (i) $\lim\phi(r)=\infty$, and (ii) for any $\epsilon>0$ there is a number K such that $\phi(x+\delta)-\phi(x)>-\epsilon$ whenever $x>K$ and $\delta>0$. The first half of the paper is devoted to showing that under these conditions, the set $\{\lambda_n\}$ can be replaced by a subsequence for which $\lim l(r)/r=0$. From this point, the method used by Boas shows the sufficiency of the conditions. The remainder of the paper is devoted to the construction of examples to show the necessity of the conditions.

R. C. Buck (Madison, Wis.).

Royster, W. C. Note on values omitted by p-valent functions. Duke Math. J. 22, 153–156 (1955).

Let $\mathfrak{F}$ denote the class of functions $f(z)=z^p+\cdots$, regular and p-valent in $|z|<1$. For $f(z)\in\mathfrak{F}$ Biernacki [Bull. Sci. Math. (2) 70, 45–51 (1946); MR 8, 326] has shown that the equation $f(z)=\alpha$ has (i) at least one root in $|z|<1$, if $|\alpha|<1/4$, and the reviewer [Tech. Rep. no. 11, Stanford Univ., Calif., 1950; MR 12, 401] that it has (ii) exactly p roots in $|z|<1$ if $|\alpha|<4^{-p}$.

The author considers the subclass of $\mathfrak{F}$ for which $|f(z)|<M$ in $|z|<1$ by putting $h(z)=f[1+f/M]^{-2}$, so that $h(z)\in\mathfrak{F}$. In this case (i) is sharpened to

$$|a|<2M^2-M-2M(M^2-M)^{1/2},$$

which is best possible. The method does not give the best possible extension of (ii) if $p>1$, despite the author's claim to the contrary, and in fact the extremals given by him in Theorems 1 and 3 are not regular throughout $|z|<1$. The correct extension of (ii) in this case is

$$|a|<[2M^{2/p}-M^{1/p}-2M^{2/p}(2M^{2/p}-M^{1/p})^{1/2}]^p$$

and may be obtained by applying the reviewer's Theorem XIII (loc. cit.) to $[f(z)]^{1/p}$. Some related results are also given. *W. K. Hayman* (Exeter).

Goodman, A. W. Almost bounded functions. Trans. Amer. Math. Soc. **78**, 82–97 (1955).

We are given $2n$ linear transformations

$$L_j(w)=\frac{a_jw+b_j}{c_jw+d_j}\quad(1\leqq j\leqq 2n),$$

with non-vanishing determinants and which are supposed to form a group $G^{(2n)}$. If w is any complex number which may be infinite, then the $2n$ values $L_j(w)$ form a set $S^{(2n)}(w)$. A function $f(z)$, meromorphic in a domain Δ, is said to be almost bounded in Δ with respect to the group $G^{(2n)}$ if, for every complex value w, $f(z)$ takes at most n values of the set $S^{(2n)}(w)$.

It is now assumed that Δ is the unit circle $|z|<1$ and that $f(0)=0$, so that $f(z)=a_1z+a_2z^2+\cdots$ near $z=0$. In the case of the group G^2 formed by w and $1/w$, $f(z)$ will be regular in $|z|<1$. In this case it is known that $|a_n|\leqq 1$ with equality only if $f(z)=\eta z^n$, $|\eta|=1$ [Milin and Lebedev, Dokl. Akad. Nauk SSSR (N.S.) **67**, 221–223 (1949); Mat. Sb. N.S. **28(70)**, 359–400 (1951); MR **11**, 339; **13**, 640]. In the general case of a $G^{(2n)}$ this result cannot be expected and, in fact, an example to the contrary is given. The author gives, however, sufficient conditions on G for $|a_1|\leqq 1$ to hold (with equality for $f(z)=\eta z$), but he requires also the additional assumption that $f(z)$ be schlicht in $|z|<1$. Amongst the groups $G^{(2n)}$ covered by these conditions are representatives of the equivalence classes of all the finite groups of linear transformations. *W. W. Rogosinski.*

Rudin, Walter. Some theorems on bounded analytic functions. Trans. Amer. Math. Soc. **78**, 333–342 (1955).

If D is any domain (i.e., connected open subset of the Riemann sphere), let $B(D)$ denote the ring of bounded analytic functions on D. A boundary point x of D is called essential if there is an $f\in B(D)$ that cannot be extended analytically to any neighborhood of D. If every boundary point of D is essential, then D is called a maximal domain. The author gives a proof that if D is maximal, then $B(D)$ determines D to within a conformal transformation. This theorem, which is due to Chevalley and Kakutani, is stated, but not proved, in a footnote in a paper of L. Bers [Bull. Amer. Math. Soc. **54**, 311–315 (1948); MR **9**, 575]. The present author's proof is both simpler (according to Kakutani) and more function-theoretic in character than the original.

Moreover, each domain D is contained in a unique maximal domain D^* (the smallest maximal domain containing D) such that each $f\in B(D)$ has an analytic extension to D^*. Thus, $B(D)$ determines D to within a conformal transformation.

If D is a maximal domain, there is an $f\in B(D)$ which has D as its natural boundary. If x is an essential boundary point of a domain D, there is an $f\in B(D)$ whose cluster set at x is the closed unit disc. Finally, an example is given of a domain D with an essential boundary point x, and an arc L contained in D except for an endpoint at x, such that for every $f\in B(D)$, $f(z)$ converges as z approaches x along L, but $f(z)$ does not approach its maximum thereon. *M. Henriksen.*

Pratje, Ilse. Iteration der Joukowski-Abbildung und ihre Streckenkomplexe. Mitt. Math. Sem. Giessen no. **48**, i+54 pp. (1954).

It was shown by Koebe [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **89**, 173–204 (1937)] that the functions uniformizing a completely ramified Riemann surface of parabolic type can be generated by the iteration of certain simple square-root transformations, which essentially reduce to $w\to z$ with (*) $w=z+z^{-1}$. The inverse function can therefore be built up by iterating the rational function (*), and the author carries this out for the case of a class of Riemann surfaces all of whose branch-points lie above p points ($p=2, 3, 4$). In spite of the simplicity of the basic transformation (*), the formulas become quite complicated even in very regular cases—such as the surfaces belonging to the exponential function and the Weierstrass $\wp$-function—and the author attempts to bring some order into the chaos by the use of a continued-fraction formalism. A much clearer picture of the steps involved in these approximations is given by means of graphs, discussed in great detail, which describe the intermediate Riemann surfaces. *Z. Nehari* (Pittsburgh, Pa.).

***af Hällström, Gunnar. Über einige Einschnittgebiete allgemeinerer Art.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 95–100 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

In a number of earlier papers [quoted in Math. Scand. **1**, 131–136 (1953); MR **15**, 208] the author considered the conformal mapping onto the unit circle of domains obtained by applying to the unit circle a countable number of radial slits. In the present paper these investigations are extended to the more general case in which the domains in question are bounded by Jordan curves to which a countable number of slits along Jordan arcs are applied. *Z. Nehari.*

Garnier, René. Sur la variation de la représentation conforme d'un domaine variable. Rend. Mat. e Appl. (5) 14, 258–267 (1954).

Let C be a Jordan curve, with interior D, in the z-plane, having bounded mean curvature, i.e., if C is given by $x=x(s)$, $y=y(s)$, then there exists a constant A such that for every h and s,

$$|x'(s+h)-x'(s)|<A|h|,\quad |y'(s+h)-y'(s)|<A|h|.$$

It is assumed further that there exists inside D a second Jordan curve C_1, with interior D_1, also having bounded mean curvature, with the property that to each point M_1 on C_1 there is a point M on C such that the normal to C at M passes through M_1 and such that 1) the distance $MM_1<k_1\epsilon$ and 2) the normal to C_1 at M_1 makes an angle of opening less than $k_2\epsilon$ with MM_1, where k_1 and k_2 are constants independent of ϵ. Let $f(z)$ and $f_1(z)$, $f(z_0)=f_1(z_0)=0$, $f'(z_0)>0$, $f_1'(z_0)>0$, be the functions which map D and D_1 respectively, onto $|Z|<1$. The author shows how the varia-

tion formula of Julia [Ann. Sci. Ecole Norm. Sup. (3) **39**, 1–28 (1922)]

$$\frac{\delta f(z)}{f(z)}=\frac{1}{2\pi i}\int_C \frac{f(z)+f(t)}{f(z)-f(t)}\left[\frac{f'(t)}{f(t)}\right]^2 \delta z dt,\quad z \varepsilon D_1,\quad \delta f=f_1(z)-f(z),$$

follows from theorems of Seidel [Math. Ann. **104**, 182–243 (1931)] and Carathéodory [S.-B. Preuss. Akad. Wiss. **1929**, 39–54]. Extensions are made to the case that C_1 and C intersect and to the case that C depends on an additional parameter. *A. J. Lohwater* (Helsinki).

Agmon, Shmuel. A property of quasi-conformal mappings. J. Rational Mech. Anal. **3**, 763–765 (1954).

The author proves the following theorem: Let

$$\zeta=\zeta(z)=\xi+i\eta$$

be a univalent mapping of the unit disc $|z|<1$. Assume that ξ and η are continuously differentiable and the Jacobian $\xi_x\eta_y-\xi_y\eta_x>0$. Suppose further that the eccentricity of the mapping

$$e=\frac{\xi_x^2+\xi_y^2+\eta_x^2+\eta_y^2}{2(\xi_x\eta_y-\xi_y\eta_x)}$$

is uniformly bounded. Then the $\lim_{r\to 1}\zeta(re^{i\omega})$ exists and is finite for almost all ω and the limiting function $\zeta(e^{i\omega})$ is not constant on any set of positive linear measure. The theorem generalizes a well-known theorem on schlicht functions to quasi-conformal mappings. The proof is quite elementary. Use is made of a theorem by L. Bers [same J. **3**, 767–787 (1954); MR **16**, 707]. *C. Loewner* (Stanford, Calif.).

Ohtsuka, Makoto. On asymptotic values of functions analytic in a circle. Trans. Amer. Math. Soc. **78**, 294–304 (1955).

The author defines several types of boundary cluster sets for functions analytic in U: $|z|<1$ which take values on an abstract Riemann surface $\mathfrak{R}$, and establishes relations between the cluster sets and the asymptotic values of the functions. A boundary point of $\mathfrak{R}$ is defined as a filter in a class $\mathfrak{F}$ in which each filter has a nested base consisting of open sets of $\mathfrak{R}$ which have no accumulation points on $\mathfrak{R}$; the set of all boundary points of $\mathfrak{R}$ is denoted by $\mathfrak{F}_{\mathfrak{R}}$. For any set E in $|z|<1$ and any point z_0 of C: $|z|=1$, the cluster set $S_{z_0}^{(E)}$ at z_0 along E is defined to be the set of all values P of $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$ such that there exists a sequence $\{z_n\}$ of E, $z_n\to z_0$, with $f(z_n)\to P$. When E is the circle $|z|<1$ or the radius terminating at z_0, the cluster sets are denoted by S_{z_0} and T_{z_0}, respectively. Let $\{K_\alpha\}$ be an open base (generally non-denumerable) of $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$; a new sequence $\{K_\alpha^*\}$ is defined by setting $K_\alpha^*=K_\alpha$ if there exists at least one open arc C_α of C containing z_0 such that the linear measure of the set $\{z \varepsilon C_\alpha;\, z\neq z_0;\, T_z\cap K_\alpha\neq\phi\}$ is zero, otherwise $K_\alpha^*=\phi$. The set $ST_{z_0}=\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}-\bigcup_\alpha K_\alpha^*$ is closed in $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$ and is regarded as a boundary cluster set. The range of $f(z)$ at z_0 is denoted by R_{z_0}.

The first of the principal results is that if $f(z)$ is analytic in $|z|<1$ and takes values on an abstract Riemann surface $\mathfrak{R}$ (with boundary $\mathfrak{F}_{\mathfrak{R}}$ if $\mathfrak{R}$ is open), then a point P_0 in $S_{z_0}-ST_{z_0}-R_{z_0}$ is an asymptotic value either at z_0 or at points z_n of $|z|=1$ converging to z_0 provided there exists a path in $\mathfrak{R}\cap S_{z_0}$ converging to P_0. Furthermore, a point P_0 in $ST_{z_0}-R_{z_0}$ is an asymptotic value of $f(z)$ if there exists a $\rho>0$ and a path in $\mathfrak{R}\cap(S_{z_0}-M_\rho^*)$ converging to P_0 (where M_ρ^* is the closure of the intersection of certain ST_z sets) and if the set on $|z|=1$ where the T_z do not contain P_0 is everywhere dense in an arc containing z_0. In the case that $f(z)$ is of bounded characteristic and R_{z_0} does not contain the values w_0, w_1, then w_0 is a radial limit of $f(z)$ near z_0 if $w_0 \varepsilon S_{z_0}-ST_{z_0}$. These results contain several theorems of the reviewer [C. R. Acad. Sci. Paris **237**, 16–18 (1953); Duke Math. J. **19**, 243–252 (1952); MR **15**, 517; **14**, 34]. Further results on asymptotic values are obtained by refining the definitions of boundary cluster sets. *A. J. Lohwater.*

Fekete, Michael. On the semi-continuity of the transfinite diameter. Bull. Res. Council Israel **3**, 333–336 (1954).

Let $\tau(D)$ be the transfinite diameter of the point set D in the z-plane. Theorem. Let Δ be a domain containing ∞ such that the complement D of $\bar{\Delta}$ consists of k simply connected domains $D_1, \cdots, D_k$. Given $\epsilon>0$, there is a lemniscate L: $|c_nz^n+c_{n-1}z^{n-1}+\cdots+c_0|=1$ consisting of branches $L_1, \cdots, L_k$, $L_j\subset D_j$, such that $1<\tau(\bar{D})/\tau(L)<1+\epsilon$. *W. H. J. Fuchs* (Ithaca, N. Y.).

Geronimus, Ya. L. On some properties of functions continuous in a closed circle. Dokl. Akad. Nauk SSSR (N.S.) **98**, 889–891 (1954). (Russian)

Let $\phi(z)=\phi(re^{i\theta})$ be continuous for $r\leqq 1$ and $\omega(\delta;\phi)$ be its m.c. (modulus of continuity) for $r=1$; Λ is the class of such functions for which $\int_0^{\pi}\omega(x;\phi)x^{-1}\,dx<\infty$. Some of the results are as follows. If $f(z)$ maps conformally a domain, bounded by a smooth closed Jordan curve C, upon $|z|<1$ and if $\theta(s)$ (angle between the axis of reals and the tangent to C at the point, whose arc coordinate is s) is in Λ, then, on $|z|=1$, $f^{(1)}(z)$ and $\lg f^{(1)}(z)$ have m.c.

$$\omega(\delta)\leqq\omega(\delta;\theta)+\omega_0(\delta),$$

where

$$\omega_0(\delta)\leqq C_1\int_0^{\delta}\omega(x;\theta)x^{-1}dx+C_2\delta\int_\delta\omega(x;\theta)x^{-2}dx.$$

If $f(z)$ is regular for $|z|<1$, is continuous for $|z|\leqq 1$ and has on $|z|=1$ m.c. $\omega(\delta)$, then

$$(1-r)|f^{(1)}(z)|\leqq C\omega\left[(1-r)\lg\frac{c}{1-r}\right]\quad(|z|=r<1)$$

and

$$|f(z_1)-f(z_2)|\leqq C\omega\left[|z_1-z_2|\lg\frac{c}{|z_1-z_2|}\right]$$

in the closed circle. The above connects with papers of J. L. Walsh and H. M. Elliot [Proc. Nat. Acad. Sci. U. S. A. **38**, 1058–1066 (1952); MR **14**, 741] and S. Warschawski [Math. Z. **35**, 321–456 (1932)]. *W. J. Trjitzinsky.*

Tumarkin, G. C. Conditions for convergence of boundary values of a sequence of analytic functions which use convergence of the moduli. Dokl. Akad. Nauk SSSR (N.S.) **98**, 739–741 (1954). (Russian)

Theorem. Let $f_n(z)$ be regular in $|z|<1$, $|f_n(z)|<M$. Necessary and sufficient conditions for the convergence in measure of the boundary functions $f_n(e^{i\theta})$ on a subset E of positive measure of $|z|=1$ are 1) $f_n(z)\to f(z)$ in $|z|<1$, and 2) $\{|f_n(e^{i\theta})|\}$ converges in measure to $|f(e^{i\theta})|$ on E. The example $f_n(z)=z^n$ shows that 2) cannot be relaxed to "$\{|f_n(e^{i\theta})|\}$ converges". Generalizations to unbounded $f_n(z)$ and to regions other than the unit circle are given. *W. H. J. Fuchs* (Ithaca, N. Y.).

Tumarkin, G. C. Approximation of functions by rational fractions with poles given beforehand. Dokl. Akad. Nauk SSSR (N.S.) 98, 909–912 (1954). (Russian)

Let C be the space of continuous functions defined on the unit-circle with the norm $\|f\|=\sup_{|z|=1}|f(z)|$; $L^p(d\sigma)$ the space of functions defined on $|z|=1$ with the norm

$$\|F\|=\left(\int_0^{2\pi}|F(e^{it})|^p d\sigma(t)\right)^{1/p}\left(d\sigma\geqq 0, \int_0^{2\pi}d\sigma<\infty\right).$$

The paper deals with the closure of rational functions with assigned poles in C and $L^p(d\sigma)$. Let A_k be a finite or infinite sequence of complex numbers, $|a|<1$ $(a \in A_k)$, B_k another sequence

$$1<|b|\leqq\infty\ (b\in B_k);\quad S_k=\sum_{a\in A_k}(1-|a|),$$
$$T_k=\sum_{b\in B_k}(1-|b|^{-1}).$$

Let $a_k(z)$ be the Blaschke product with zeros A_k;

$$\mu(z)=\limsup_{k\to\infty}\log|a_k(z)|.$$

The function $\mu(z)$ is subharmonic and not identically $-\infty$. Let $u(z)$ be the least harmonic majorant of $\mu(z)$, $v(z)$ a conjugate harmonic function of $u(z)$, $\phi(z)=\exp(\mu+iv)$. The Blaschke product $B(z)$ has r zeros at α, if for any $M>0$ and any neighborhood N of α there is a K such that for $k>K$ either $S_k>M$ or A_k has at least r points in N. It is known that the set of rational functions $\{R_k\}$ such that all poles of R_k lie in $A_k\cup B_k$ is complete in C, if and only if $S_k\to\infty$, $T_k\to\infty$ as $k\to\infty$. The author states Theorem 1. Suppose

$$(1)\qquad \liminf_{k\to\infty}S_k<\infty,\quad \lim_{k\to\infty}T_k=\infty.$$

The function $f(e^{i\theta})$ belongs to the C-closure of $\{R_k(z)\}$ if and only if $f(e^{i\theta})$ is the boundary function of (2) $f(z)=g(z)/(B(z)\phi(z))$, where $g(z)$ is regular and bounded in $|z|<1$. Theorem 2. If $S_k+T_k\to\infty$ as $k\to\infty$, and if (3) $\int_0^{2\pi}\log\sigma'(t)dt=-\infty$, then $\{R_k\}$ is fundamental in $L^p(d\sigma)$ $(p>0)$. Theorem 3. If (3) is not satisfied and (1) holds, then the closure of $\{R_k\}$ in $L^p(d\sigma)$ $(p>0)$ consists of functions of the form (2), where $g(z)$ is regular and of bounded (Nevanlinna) characteristic in $|z|<1$. Analogues are given for the spaces C and $L^p(d\sigma)$ on the infinite x-axis.

W. H. J. Fuchs (Ithaca, N. Y.).

Macintyre, Sheila Scott. Transform theory and Newton's interpolation series. Proc. London Math. Soc. (3) 4, 385–401 (1954).

Let F be entire, $h(\varphi)=\limsup r^{-1}\log F(re^{i\varphi})$ and let D be the convex set whose supporting function is $h(-\varphi)$. Then, $F(z)=(2\pi i)^{-1}\int_\Gamma e^{sz}f(s)\,ds$, where f is the Borel-Laplace transform of F, and Γ encloses D. If

$$|F(re^{i\varphi})|\leqq K(1+r)^\beta e^{rh(\varphi)},$$

Schmidli [Thesis, Zurich, 1942; MR 4, 39] proved that if $\beta<-1$, then Γ can be closed down to ∂D, the boundary of D. If $\beta<k$ where k is an integer, $k>0$, then this result may be applied to the entire function $G(z)=\{F(z)-P(z)\}/Q(z)$ where $Q(z)=z(z-1)(z-2)\cdots(z-k)$, and where P is the unique polynomial of degree k satisfying $P(j)=F(j)$, $j=0,1,2,\cdots,k$, resulting in a representation formula $F(z)=P(z)+(2\pi i)^{-1}Q(z)\int_{\partial D}e^{sz}g(s)\,ds$. The author obtains a generalization of this for k non-integral, using an extension of the Pólya representation due to A. J. Macintyre [Proc. London Math. Soc. (2) 45, 1–20 (1938). Let $F(\alpha+re^{i\varphi})$ be analytic for $|\varphi|\leqq\theta<\pi$ and there obey

$$|F(\alpha+re^{i\varphi})|\leqq K(1+r)^\beta e^{rh(\varphi)}.$$

Let μ obey $\alpha<\mu$, $\beta<\mu$, and set $k=[\mu-\alpha]$. Let P be the polynomial of degree k such that $P(\mu-j)=F(\mu-j)$ for $j=0,1,\cdots,k$. Let $q(z)=\Gamma(z+1)/\Gamma(z-\mu)$ and set

$$G(z)=\{F(z)-P(z)\}/q(z).$$

If $g(s)$ is the Laplace transform of G, then

$$F(z)=P(z)+(2\pi i)^{-1}q(z)\int_\Gamma e^{sz}g(s)\,ds,$$

where Γ is the boundary of an unbounded convex set determined by $h(\varphi)$. This result is then applied to settle a long-standing conjecture on the convergence properties of the Newton interpolation series, $\sum_0^\infty\binom{z}{n}\Delta^nF(0)$. Suppose now that the numbers α and β are negative and that $h(\varphi)\leqq\cos\varphi\log(2\cos\varphi)+\varphi\sin\varphi$. Nörlund [Leçons sur les séries d'interpolation, Gauthier-Villars, Paris, 1926] proved that the Newton series converged to $F(z)$ for all z in the half plane $x>\max\{\alpha,\beta+\frac{1}{2}\}$. When F is entire and $\beta<-1$, it is known that convergence occurs for $x>-1$ [Buck, Trans. Amer. Math. Soc. 64, 283–298 (1948); MR 10, 693]. The author now shows that in the general case, convergence occurs for $x>\max\{\alpha,\beta\}$. She also obtains an analogous result in the case where α is a positive. [The statement of Theorem 3 contains several minor misprints; in (1.06), the exponent "r" on (-1) should be deleted.]

R. C. Buck (Madison, Wis.).

Mergelyan, S. N., and Tamadyan, A. P. On completeness in a class of non-Jordan regions. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Estest. Tehn. Nauk 7, 1–17 (1954). (Russian. Armenian summary)

Let P be a perfect, nowhere dense set on the circumference $|z|=1$. Let D be the domain obtained from $|z|<1$ by excluding all points $z=re^{i\alpha}$ with $e^{i\alpha}\in P$, $0<h\leqq r<1$. The polynomials are dense in the Hilbert space of functions regular in D with the norm defined by $\|F\|^2=\iint_D|F(z)|^2dxdy$, if there is a denumerable set N dense in P such that every point of N can be included in an arbitrarily short interval Δ on $|z|=1$ satisfying

$$(1)\qquad m(\Delta\cap CP)<\exp(-\exp 31h/m(\Delta)),$$

where $m(S)$ is the linear measure of the set S. The term $m(\Delta)$ cannot be replaced by $m^\alpha(\Delta)$, $\alpha<1$.

W. H. J. Fuchs (Ithaca, N. Y.).

Sakakihara, Kanenji. Meromorphic approximations on Riemann surfaces. J. Inst. Polytech. Osaka City Univ. Ser. A. 5, 63–70 (1954).

In 1926 Walsh [Math. Ann. 96, 430–450 (1926)] proved the following theorem for domains of the complex z-plane which are bounded by one Jordan curve: If a function $f(z)$ is holomorphic in the interior of D and continuous in the corresponding closed region $\bar{D}$, then in $\bar{D}$ the function $f(z)$ can be uniformly approximated by a sequence of polynomials. The author extends this theorem to domains D, which are part of Riemann surfaces and which are bounded by a finite number of Jordan curves. Specifically he shows that in $\bar{D}$ every function $f(z)$ which is holomorphic in D and continuous in $\bar{D}$ can be uniformly expanded in a series of functions $h(z)$ meromorphic in R. Therein $h(z)$ has one and only one pole in every connected compact component of $R-\bar{D}$. A similar statement holds for the case that $f(z)$ has infinitely many poles in the interior of D.

H. Grauert (Münster).

Lehto, Olli. Value distribution and boundary behaviour of a function of bounded characteristic and the Riemann surface of its inverse function. Ann. Acad. Sci. Fenn. Ser. A. I. no. **177**, 46 pp. (1954).

Let $f(z)$ be meromorphic with bounded characteristic in $|z|<1$; then the function $N(r, a)=\int_0^r r^{-1}n(r, a)\,dr$ has the property $N(a)=\lim_{r\to 1} N(r, a)<\infty$. It is assumed that, for almost all $e^{i\varphi}$ on $|z|=1$, the radial limit values $f(e^{i\varphi})$ lie in a given closed set Γ whose complement is not empty. The problem studied in the first part of the paper is the behavior of $N(a)$ as a function of a in a component D of the complement of Γ; the result, obtained by the author in earlier papers [same Ann. no. **160** (1953); Acta Math. **91**, 87–112 (1954); MR **15**, 517, 947], is that $N(a)$ is equal to its least harmonic majorant $P(a)$ except on a possible set of capacity zero. A value a is called normal for $f(z)$ if $P(a)=N(a)$, exceptional if $P(a)>N(a)$. The second problem considered is that of the boundary behavior of $f(z)$ whenever $f(z)$ takes a value in D relatively seldom. To attack this problem, the author introduces to his class of functions Iversen's notion of direct and indirect critical singularities of the inverse function $z=f^{-1}(w)$. It is shown that a direct critical singularity outside Γ is always exceptional, and that there exist indirect critical singularities of $f^{-1}(w)$ which are normal for $f(z)$. The author also proves, as in his cited papers, that an exceptional value is always an asymptotic value. One of the principal results is the following: if $w=a$ is not in Γ and if D is the component containing $w=a$, then either 1) if $f(z)$ takes one value in D infinitely often, every other value in D is taken infinitely often except for a possible set of capacity zero; or else 2) if $f(z)$ takes the value $w=a$ finitely often and if $w=a$ is a normal value for $f(z)$, then every value in D is normal for $f(z)$ and is assumed equally often by $f(z)$. It is then shown how an extension of Lindelöf's theorem by Collingwood and Cartwright [ibid. **87**, 83–146 (1952); MR **14**, 260] for meromorphic functions omitting at least three values can improve certain of his results, e.g., if $f(z)$ is of bounded characteristic and omits at least three values in $|z|<1$ and if D is an arbitrary component in the complement of the closure of all boundary values of $f(z)$, then either $f(z)$ takes no value in D or else every value of D is a normal value of $f(z)$. A counterexample is provided for the case that $f(z)$ does not omit three values.

A. J. Lohwater (Helsinki).

***Lehto, Olli. On meromorphic functions of bounded characteristic.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 183–187 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

A summary of the author's recent results in the theory of meromorphic functions of bounded characteristic [C. R. Acad. Sci. Paris **236**, 1943–1945 (1953); Ann. Acad. Sci. Fenn. Ser. A. I. no. **160** (1953); no. **177** (1954); Acta. Math. **91**, 87–112 (1954); MR **14**, 858; **15**, 517, 947; and the preceding review]. *A. J. Lohwater* (Helsinki).

Tietz, Horst. Zur Realisierung Riemannscher Flächen. Math. Ann. **128**, 453–458 (1955).

The author introduces the following new aspect of the theory of conformal mapping. Given a Riemann surface F and a planar compact subregion $M\subset F$, do there exist meromorphic functions on F which are univalent on M. The answer turns out to be in the affirmative, regardless of the genus g of F. In particular, if g is finite and C denotes a set of g disjoint nondividing cycles, then F is conformally equivalent to a covering surface of the plane such that the image of a given $M\subset F-C$ is a plane compact region. These results are extended to p-valent functions and, more generally, to functions with any suitably restricted properties on M. The proofs are based on the Behnke-Stein theorem: any function analytic on M can be uniformly approximated by functions meromorphic on F [Math. Ann. **120**, 430–461 (1949); MR **10**, 696]. *L. Sario.*

Washnitzer, G. A Dirichlet principle for analytic functions of several complex variables. Ann. of Math. (2) **61**, 190–195 (1955).

Let V be a complex manifold of n complex dimensions. The author defines complex densities on V and an inner product and norm for densities in the usual way. Let $\mathfrak{B}^*$ be the closure of the linear space of densities with finite norm on V; $\mathfrak{B}^*$ is a Hilbert space. Let $\xi^1, \cdots, \xi^n$ be n real-valued functions on V which vanish outside a compact subset contained in a coordinate neighborhood and which have continuous partial derivatives. Then the functional determinant $\partial(\xi^1, \cdots, \xi^n)/\partial(u_1, \cdots, u_n)$ $(u_1, \cdots, u_n$ the local coordinates) is a density and called "nullifier". Let $\mathfrak{Z}^*$ be the closure of the linear space that is generated by the set of nullifiers on V; $\mathfrak{Z}^*$ is a Hilbert space and a subspace of $\mathfrak{B}^*$. The author proves: If $\mathfrak{U}^*$ is the orthogonal complement of $\mathfrak{Z}^*$ in $\mathfrak{B}^*$, then $\mathfrak{U}^*$ is the space of all bounded complex analytic densities on V. Also: If $W_0\in\mathfrak{B}^*$ and $d=\inf\|W\|$ (the infimum taken with respect to all W that have "equal boundary values" with W_0, that is $W-W_0\in\mathfrak{Z}^*$), then there exists a unique J in $\mathfrak{B}^*$ with $\|J\|=d$; J is a complex analytic density (Generalized Dirichlet principle). For the proof the author evaluates the integral that defines the scalar product for functions that are orthogonal to the nullifiers.

H. J. Bremermann (Münster).

Egesoy, E. Die unbeschränkten Hartogsschen Körper. Comm. Fac. Sci. Univ. Ankara. Sér. A. **6**, 17–31 (1954). (Turkish summary)

The author proves several results on analytic mappings of unbounded Hartogs' domains $[w\in G,\ |z|<\mathrm{Re}(w)]$ in the space of two complex variables. Some of the results are similar to results obtained by H. Behnke and E. Peschl [Math. Ann. **112**, 433–468 (1936); **114**, 69–73 (1937)] for unbounded Reinhardt and circular domains. Among the author's results is the following theorem (Theorem 2). If an unbounded proper Hartogs' domain H is a domain of regularity, then there exists at least one analytic plane $w=c$ which lies entirely in H (except possibly for the point at infinity in the plane). He also obtains the following result (Theorem 9). Let H be a Hartogs' domain in which there exist at least two regular bounded functions whose functional determinant does not vanish identically. If H is mapped in a 1-1 analytic manner on itself by means of a transformation.

$$w'=f(w, z)=\sum_{n=0}^{\infty}f_n(w)z^n;\quad z'=g(w, z)=\sum_{n=0}^{\infty}g_n(w)z^n$$

and if $f_0(w)=aw+$higher powers, $g_0(w)=0$, $g_1(w)=bw$ $+$higher powers, with $ab\neq 0$, then $f(w, z)=f_1(w)$ and $g(w, z)=g_1(w)z$. *W. T. Martin* (Cambridge, Mass.).

Bingen, Franz. Les domaines bornés symétriques de l'espace complexe à n dimensions. Bull. Soc. Math. Belgique **6** (1953), 53–61 (1954).

Texte d'une conférence où sont exposés les principaux résultats connus de la théorie des domaines bornés symé-

triques (dans l'espace de n variables complexes), théorie due essentiellement à E. Cartan [Oeuvres complètes, partie I, v. 2, Gauthier-Villars, Paris, 1952, pp. 1259–1305; MR 14, 343], avec des compléments récents dus à A. Borel et A. Lichnerowicz [voir Lichnerowicz, Géométrie differentielle, Colloques Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 171–184; MR 16, 519]. *H. Cartan.*

Berezin, F. A., and Pyateckiĭ-Šapiro, I. I. Homogeneous extensions of a complex space. Dokl. Akad. Nauk SSSR (N.S.) **99**, 889–892 (1954). (Russian)

A manifold M with m complex dimensions is called a homogeneous extension of the affine complex m-dimensional space C if the following conditions are satisfied: i) there exists an analytic mapping f of C into M, such that f is one-to-one and $f(C)$ is everywhere dense in M; ii) the group of analytic homeomorphisms of M is transitive. The author proves that the preceding set of conditions is equivalent to each of the following three sets of conditions: i) M is a homogeneous, simply connected, algebraic manifold. ii) M is homogeneous, and the field of functions meromorphic on M is isomorphic to the field of rational functions of m variables. iii) M can be described as the space of cosets of a certain complex semisimple group G with respect to a subgroup which contains the maximal solvable subgroup of G. *H. Tornehave* (Copenhagen).

Kreyszig, Erwin. Stetige Modifikationen komplexer Mannigfaltigkeiten. Math. Ann. **128**, 479–492 (1955).

Par modification $\mathfrak{M}(G^n, H^n)$ d'une variété analytique complexe G^n (sur un ensemble M) en une variété H^n, on entendra [cf. Behnke et Stein, Math. Ann. **124**, 1–16 (1951); MR 13, 644]: $G^n=A\cup M$; $H^n=B\cup N$; les ensembles A et B sont ouverts sur G^n et H^n respectivement, en correspondance pseudo-conforme $A=T(B)$; de plus à un voisinage U de l'ensemble fermé $M\subset G^n$ et à un point $Q\in N\subset H^n$, on peut associer un voisinage V sur H^n de manière que $T(V\cap B)\subset U$.

La modification $\mathfrak{M}(G^n, H^n)$ est dite continue, analytique, propre, si T se prolonge en une représentation $G=T(H)$ ayant l'un de ces propriétés. Si M est un sous-ensemble analytique de G^n et si $m(G^n, H^n)$ est continue, elle est analytique; N est alors un sous-ensemble analytique de H^n; à une fonction g holomorphe (méromorphe) sur G^n correspond une fonction γ holomorphe (méromorphe) sur H^n avec $\gamma=g\circ T$ sur B; à un ensemble analytique α sur G^n correspond un ensemble analytique β sur H^n avec $T[\beta\cap B]=\alpha\cap A$; $T(\beta)$ est aussi un ensemble analytique sur G^n si T est propre [cf. R. Remmert, Dissertation, Münster, 1954]. Si $\mathfrak{M}(G^n, H^n)$ est non triviale, N est un ensemble analytique de dimension $n-1$; si de plus la modification est propre, T se prolonge sur H^n: $G^n=T(H^n)$, M est un ensemble analytique, et N est irréducible si M l'est. Dans le cas où M et N sont irréductibles et de même dimension, la modification est nécessairement triviale.

On donne ensuite une extension $\sigma^{n,k}$ de l'éclatement de H. Hopf: $\sigma^{n,k}$ consiste en une modification de G^n sur une sous-variété analytique $M=N^k$, $0\leqq k\leqq n-1$, n'ayant que des points ordinaires: on substitue à N^k un espace fibré obtenu en faisant éclater les points de N^k dans des sous espaces de dimension $n-k$: $\sigma^{k,0}$ est l'éclatement de Hopf d'un point d'une variété G^k en un espace projectif complexe P^{k-1}; pour $k>0$, le procédé étant local, on définit $\sigma^{n,k}$, en se ramenant au cas où N^k est la variété $z_1=z_2\cdots=z_{n-k}=0$ au voisinage de l'origine et on fait éclater les points de N^k dans l'espace C^{n-k} $(z_1, \cdots, z_{n-k})$ en des espaces P^{n-k-1}; on établit que $\sigma^{n,k}$ est indépendant du choix des coordonnées locales z_i et est une modification $\mathfrak{M}(G^n, H^n)$. *P. Lelong* (Paris).

Stoll, Wilhelm. Über meromorphe Modifikationen. I. Allgemeine Eigenschaften der Modifikationen. Math. Z. **61**, 206–234 (1954).

La définition adoptée pour une modification analytique se rattache aux travaux de H. Hopf [Rend. Mat. e Appl. (5) **10**, 169–182 (1951); MR 13, 861] et diffère de celle de Behnke et Stein rappelée plus haut. Une modification notée $\mathfrak{M}(G, A, M, \tau; H, B, N, \upsilon)$ comprend: a) deux variétés analytiques complexes G et H de dimension complexe n; b) un sous-ensemble fermé M de G, $M\subset\lambda\subset U\subset G$, λ étant un ensemble analytique dans G, U un voisinage ouvert de M sur G; c) symmétriquement sur H: $N\subset\mu\subset V\subset H$, N étant fermé et μ un ensemble analytique dans H, V un voisinage ouvert de N sur H; d) une transformation pseudo-conforme τ: $A=G-M\rightarrow B=H-N$; on pose $\upsilon=\tau^{-1}$.

La modification inverse est

$$\mathfrak{M}^{-1}=\mathfrak{M}(H, B, N, \upsilon; G, A, M, \tau);$$

on écrit l'équivalence $\mathfrak{M}_1\sim\mathfrak{M}_2$ si on passe de la modification $\mathfrak{M}_1$ à $\mathfrak{M}_2$ par des transformations pseudo-conformes α, β:

$$\alpha(G_1)=G_2,\quad \alpha(A_1)=A_2,\ \cdots,\beta(H_1)=H_2,\ \cdots,\beta^{-1}\tau_2\alpha=\tau_1.$$

Les variétés G, H ne sont pas supposées en général à base dénombrable. Une modification est dite ouverte si, pour chaque ensemble ouvert $U\supset M$, $\tau(U\cap A)\cup N$ est ouvert; il faut et il suffit pour qu'il en soit ainsi que pour toute suite $Q_2\in B$, avec $\lim Q_2=Q\in N$, $P_2=\upsilon(Q_2)$ ait au moins un point d'accumulation dans M. Etude des ensembles d'éclatement $\Sigma(R, L)$ sur H, de τ le long de L dans R $(R\subset M, L\subset G)$: $Q\in H$ appartient à $\Sigma(R, L)$ s'il existe une suite $P_2\in L\cap A$ avec $Q=\lim\tau(P_2)$, $P=\lim P_2\in R$. Si $\mathfrak{M}$ est ouverte, R étant formé à base dénombrable, $\Sigma(R, L)$ est fermé. Les fonctions analytiques dans un voisinage U de M sur G se transportent par $g(Q)=f[\upsilon(Q)]$ sur $\tau(U\cap A)$ dans H et se prolongent sur $V=\tau(U\cap A)\cup N$; sous une condition supplémentaire on peut faire le prolongement d'une fonction méromorphe sur U; de même celui d'un ensemble analytique L, quand $L\cap M$ consiste en un nombre fini de points en lesquels on suppose que τ n'éclate pas. Le travail (qui annonce d'autres résultats à paraitre, notamment qu'une $\mathfrak{M}$ ouverte et méromorphe entre des variétés G et H compactes se ramène à des éclatements successifs de Hopf) se termine par une étude des transformées, dans une modification $\mathfrak{M}$ ouverte, de courbes analytiques complexes L^1, sous des hypothèses restrictives concernant l'ensemble $L^1\cap M$. *P. Lelong* (Paris).

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. On the Schwarzian lemma in the matrix space. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **4**, 309–316 (1954).

Denote by R the linear space of all matrices with m rows and n columns (type (m, n)), the elements being complex numbers. The norm of $A\in R$ is defined as the norm of the linear operator corresponding to A, which maps complex Euclidean n-space into complex Euclidean m-space. Let $Z=(z_{ij})$ be an independent variable ranging over R. A function W on R into R is called analytic if the elements of the matrix $W(Z)$ are analytic functions of the mn variables $z_{11}, \cdots, z_{mn}$. M. Sugawara [Proc. Imp. Acad. Tokyo **17**, 483–488 (1941); MR 7, 380] showed that the classical formal statement of Schwarz's lemma remains valid, i.e. $\|W(Z)\|\leqq\|Z\|$ when $\|Z\|\leqq 1$, provided that W is analytic

and $\|W(Z)\| \leq 1$ when $\|Z\| \leq 1$, and $W(0)=0$. In the present paper the condition $W(0)=0$ is replaced by $W(A)=B$, where $\|A\|<1$ and $\|B\|<1$, and a more complicated inequality for $W(Z)$ is deduced. The result depends on a theorem about when a function of the form

$$W=(A_1Z+A_2)(A_3Z+A_4)^{-1}$$

is a one-to-one mapping of $\|Z\| \leq 1$ onto $\|W\| \leq 1$, where A_1, A_2, A_3, A_4 are of types (m,n), (m,n), (n,m), (n,n) respectively. *A. E. Taylor* (Los Angeles, Calif.).

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. On extension of Schwarzian lemma in matrix space. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 4, 317–318 (1954).

The notations and definitions are the same as in the preceding review. Let W be as in the Sugawara form of Schwarz's lemma, and suppose further that $W(tZ)$ has the form $t^mG(t,Z)$, where m is a non-negative integer and G is analytic as a function of t when $\|tZ\| \leq 1$, with $G(0,Z)=0$. Then $\|W(Z)\| \leq \|Z\|^{m+1}M$, where $M=\max \|W(Z)\|$ when $\|Z\|=1$. *A. E. Taylor* (Los Angeles, Calif.).

Elianu, I. P. Sur les fonctions non analytiques à plusieurs variables complexes. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 511–521 (1954). (Romanian. Russian and French summaries)

The author considers functions

$$f(z)=X(x_1, y_1, \cdots, x_n, y_n)+iY(\cdots),$$

where $z=(z_1, \cdots, z_n)$, $z_j=x_j+iy_j$ is in the complex plane P_j, the values of f are in a complex plane P. Set functions of the form $F(E)=X(E)+iY(E)$ are considered; here $E=(E_1, \cdots, E_n)=\{z_1 \in E_1, \cdots, z_n \in E_n\}$, the E_j being bounded L-measurable. $F(E)$ is denumerably additive, is continuous or absolutely continuous (relative to E_j) in a finite domain Δ_j or on a set M_j, if

$$\Phi_0(E_j)=F(z_1^0, \cdots, z^0_{j+1}, E_j, z^0_{j+1}, \cdots, z_n^0)$$

has the same property relative to every $z_k^0 \in E_k$ $(k\neq j)$. Let ω_j be a square of center z_j^0. One defines the extreme limits of $\Phi(\omega_j)/\text{meas}\ \omega_j$ (when meas $\omega_j \to 0$), the extreme symmetric derivatives of $F(E)$ relative to E_j and at $z_j=z_j^0$, as well as the extreme partial derivatives (related to regular sequences of sets). It is assumed that the function $\Phi(\delta_j)=(2i)^{-1}\int_{\gamma_j} f(z)\,dz$ (γ_j being the positively described contour limiting a domain $\delta_j \in P_j$) is absolutely continuous in δ_j; one defines the areal partial derivative $Df/D\omega_j$ relative to z_j. A formula is established, involving multipole integration and derivatives of the indicated kind; this leads to the main result, consisting of formulas representing $f(z)$ in terms of repeated integrals involving Cauchy kernels and suitable areal derivatives. The conditions involved entail generalization of the known Pompeiu-Teodorescu formulas for the case of a single complex variable (the functions are generally non-analytic). *W. J. Trjitzinsky* (Urbana, Ill.).

Theory of Series

***Hardi, G. [Hardy, G. H.] Rashodyaščiesya ryady. [Divergent series.]** Translated by D. A. Raĭkov, with a preface and survey article by S. B. Stečkin. Izdat. Inostrannoi Literatury, Moscow, 1951. 504 pp. 29.50 rubles.

This is a literal translation from English to Russian of the well-known textbook and reference book on series by G. H. Hardy [Divergent series, Oxford, 1949; MR 11, 25]. The only significant difference between the two editions is an appendix in the Russian edition by S. B. Stečkin. This appendix gives, with 19 appropriate references, an excellently motivated exposition of the Bernstein-Rogosinski trigonometric methods for evaluation of series. A real sequence $\alpha_0, \alpha_1, \cdots$ determines a method (BR, α_n) by means of which a series $\sum u_n$ is evaluable to s if

$$\lim_{n\to\infty} \sum_{k=0}^{n} (\cos k\alpha_n)u_k=s.$$

An interesting feature of this appendix is the presentation and use, without historical remarks, of a theorem giving sufficient conditions that a matrix method for evaluation of series be equivalent to convergence. A more general version of this theorem is given, with historical remarks, by Agnew [Proc. Amer. Math. Soc. 3, 550–556 (1952); MR 14, 39], and applications to Bernstein-Rogosinski methods beyond those included in the appendix under review are given by Petersen [Canad. J. Math. 4, 445–454 (1952); MR 14, 368] and Agnew [Ann. of Math. (2) 56, 537–559 (1952); MR 14, 368]. *R. P. Agnew* (Ithaca, N. Y.).

Siddons, A. W. The product of two series. Math. Gaz. 39, 4–6 (1955).

Copping, J. K-matrices which sum no bounded divergent sequence. J. London Math. Soc. 30, 123–127 (1955).

An infinite matrix A is called a K_r-matrix whenever it has bounded row-norms; it is called a K-matrix if it transforms every convergent sequence into a convergent sequence. The author proves the following theorem: If the K-matrix A has a left-hand reciprocal C which is a K_r-matrix, then A evaluates no bounded divergent sequence. He compares his result with those of A. Wilansky [Bull. Amer. Math. Soc. 55, 914–916 (1949); MR 11, 243], R. P. Agnew [Proc. Amer. Math. Soc. 3, 550–556 (1952); MR 14, 39], A. M. Tropper [ibid. 4, 671–677 (1953); MR 15, 118] and C. F. Martin [Proc. Amer. Math. Soc. 5, 863–865 (1954); MR 16, 351]. In the paper reviewed below A. Wilansky and the reviewer give necessary and sufficient conditions for a K-matrix to sum a bounded divergent sequence. *K. Zeller* (Tübingen).

Wilansky, Albert, and Zeller, Karl. Summation of bounded divergent sequences, topological methods. Trans. Amer. Math. Soc. 78, 501–509 (1955).

The authors discuss conservative methods of summation A with the rather pathological property that they do not sum any bounded divergent sequence. Let c, m, c_A, m_A be the spaces of convergent, bounded, A-convergent and A-bounded sequences, respectively. With a natural sequence of pseudo-norms, c_A and m_A are spaces of type (F). The main results are as follows: The conditions: (a) c is closed in c_A; (b) m is closed in m_A; (c) A sums no bounded divergent sequences, are equivalent. Let the sequences s^k, $k=1, 2, \cdots$, be linearly independent modulo m, so that no non-trivial linear combination $a_1s^1+\cdots+a_ks^k$ is bounded and let r be a positive integer. Then there exist regular row-finite methods A, B so that c_A is the linear hull of c, $s^1, \cdots, s^r$; and c_B contains s^k, $k=1, 2, \cdots$, but no bounded divergent sequences. *G. G. Lorentz* (Detroit, Mich.).

Newton, R. H. C. On the summation of periodic sequences. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 533–544, 545–549 (1954).

The author investigates the summability of periodic sequences. Results are obtained for the case of a general matrix method and also for the special methods of Nörlund, Abel, Cesàro, Hölder, and Riesz. The reviewer observes that certain of the results could be obtained more directly by application of known theorems. For example, it is evident that every periodic sequence is bounded and summable-$(C, 1)$ and therefore summable-$(C, \alpha>0)$; the latter implies Hölder and Riesz summability as well as Abel summability to what the author hopefully calls the "right" value.

J. D. Hill (East Lansing, Mich.).

Rajagopal, C. T. Theorems on the product of two summability methods with applications. J. Indian Math. Soc. (N.S.) **18**, 89–105 (1954).

The author proves that AB-summability of a sequence s_n is implied by its A-summability in case when B is a regular Hausdorff method and the A-summability is defined by the relation

$$\sum_0^\infty p_n x^n s_n / \sum_0^\infty p_n x^n \to s, \quad x \to \rho - 0,$$

and some additional conditions which cannot be reproduced here. In case when A is the Abel or the Borel transform, this was given by O. Szász [Ann. Soc. Polon. Math. **25**, 75–84 (1953); MR **15**, 26]. There is also a continuous analogue of the above theorem and applications to Tauberian theorems, one of which is as follows: If $s(x)$ is A-summable to s and

$$C_k(x) - C_{k+1}(x) \geqq -M,$$

where

$$C_k(x) = kx^{-k} \int_0^x (x-u)^{k-1} s(u)du, \quad k \geqq 0,$$

then $C_{k+1}(x) \to s$ for $x \to \infty$.

G. G. Lorentz.

Tanzi Cattabianchi, Luigi. Perturbazione media-ereditaria e limiti delle successioni. Riv. Mat. Univ. Parma **5**, 125–136 (1954).

The main results are closely related to Mercerian theorems of Vijayaraghavan [J. London Math. Soc. **3**, 130–134 (1928)]. Let x_n be a real sequence and let

$$v_n = x_n + \beta_n (x_0 + x_1 + \cdots + x_n)/(n+1).$$

If $v_n \to 0$ and if there exist positive constants A and B and a positive monotone decreasing sequence $\psi(n)$ such that $A\psi_n < \beta_n/n < B\psi_n$, then $x_n \to 0$. To each sequence $\epsilon_0, \epsilon_1, \cdots$ for which $\epsilon_n > 0$ and $\epsilon_n \to 0$ there correspond real sequences x_n and β_n for which $\beta_n/n > \epsilon_n$, $v_n \to 0$, and $\limsup |x_n| = \infty$.

R. P. Agnew (Ithaca, N. Y.).

Erdös, Paul, Herzog, Fritz, and Piranian, George. Sets of divergence of Taylor series and of trigonometric series. Math. Scand. **2**, 262–266 (1954).

Eine Menge E auf C: $|z|=1$ heisst F_σ-Menge, wenn $E = \bigcup_{n=1}^\infty A_n$ (A_n abgeschlossen); sie heisst G_δ-Menge, wenn $E = \bigcap_{n=1}^\infty O_n$ (O_n offen); sie heisst vom logarithmischen Mass 0, wenn zu jedem $\epsilon>0$ eine Überdeckung von E durch abzählbar viele Kreisbögen γ_n mit den Längen L_n (<1) existiert, für die $\sum_{n=1}^\infty 1/|\log L_n| < \epsilon$ ist. Die Verfasser setzen frühere Untersuchungen über Divergenzpunktmengen fort [Herzog und Piranian, Duke Math. J. **16**, 529–534 (1949); **20**, 41–54 (1953); MR **11**, 91; **14**, 738]. 1. Zu jeder Menge E auf C vom logarithmischen Mass 0 [und vom Typ F_σ] gibt es ein $f(z) = \sum a_n z^n$ mit Teilsummen $s_n(z)$ und mit folgenden Eigenschaften: (i) $f(z)$ ist in $|z| \leqq 1$ stetig; (ii) $\sum a_n z^n$ divergiert auf E [und konvergiert auf $C-E$]; (iii) $\{s_n(z)\}$ ist gleichmässig beschränkt auf C. 2. Ist E_1 eine F_σ-Menge vom logarithmischen Mass 0 und E_2 eine G_δ-Menge, so gibt es eine Potenzreihe $\sum a_n z^n$, die auf $E = E_1 + E_2$ divergiert, auf $C-E$ aber konvergiert. Ganz entsprechende Resultate gelten für Fourier-Reihen von Funktionen, die in $\langle 0, 2\pi \rangle$ stetig sind, bzw. für trigonometrische Reihen; dadurch werden Resultate von Tandori [Publ. Math. Debrecen **2**, 191–193 (1952); MR **14**, 745] verbessert. Beweishilfsmittel sind u.a. die Fejérschen Polynome

$$P_n(z) = \frac{1}{n} + \frac{z}{n-1} + \cdots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \frac{z^{n+1}}{2} - \cdots - \frac{z^{2n-1}}{n}.$$

D. Gaier (Stuttgart).

Aljančić, S. Développement asymptotique des fonctions représentables par les séries de Legendre. Acad. Serbe Sci. Publ. Inst. Math. **6**, 115–124 (1954).

Proof of the following theorem. Suppose $0 < \epsilon \leqq \theta \leqq \pi - \epsilon$ and let x be an unbounded variable. If the numbers $a_0, a_1, \cdots$ are independent of x and form a completely monotonic sequence, if the numbers $b_0, b_1, \cdots$, depending on x, possess the property that $\sum_{\gamma=k+1}^\infty \gamma^{-1/2} |\Delta^{k+1} b_{\gamma-k-1}|$ tends for large $|x|$ asymptotically to zero as $k \to \infty$ and finally, if it is possible to find polynomials $p_{\gamma 0}, p_{\gamma 1}, \cdots$ in γ whose coefficients depend on x such that $p_{\gamma h}$ tends for each constant integer $\gamma \geqq 0$ and for large $|x|$ asymptotically to b_γ as $h \to \infty$, then the Abel-Poisson sum of the series $\sum a_\gamma b_\gamma P_\gamma(\cos\theta)$ exists and is, uniformly in θ, an asymptotic limit of $\sum a_\gamma p_{\gamma h} P_\gamma(\cos\theta)$ as h approaches infinity.

J. G. van der Corput (Berkeley, Calif.).

Fourier Series and Generalizations, Integral Transforms

Pyateckiĭ-Šapiro, I. I. Supplement to the work "On the problem of uniqueness of expansion of a function in a trigonometric series." Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 79–97 (1954). (Russian)

The paper mentioned in the title of the present one appeared in Moskov. Gos. Univ. Uč. Zap. **155**, Mat. **5**, 54–72 (1952), and has so far not been available for review. The earlier paper contains some errors which are corrected in the present article. The main results are as follows. (1) A closed set $E \subset (0, 2\pi)$ is called a set of multiplicity, if there is a trigonometric series S converging to 0 outside E but not everywhere. If S is, in addition, the Fourier-Stieltjes series of a positive mass distribution concentrated on E, E is called a set of multiplicity in the restricted sense. The author gives a new proof of the result (formulated in the previous paper) that there are sets of multiplicity which are not of multiplicity in the restricted sense. (2) Call C the class of all algebraic integers $\theta>2$ whose conjugates other than θ have moduli less than 1. Denote by $E(\theta)$ the perfect set constructed on $(0, 2\pi)$ in the familiar Cantor fashion, except that at each stage we remove not the middle third but a concentric interval of length $1-2/\theta$. Salem showed that if θ is not in C, then $E(\theta)$ is a set of multiplicity, and conjectured that if $\theta \in C$, then $E(\theta)$ is a set of uniqueness [see Trans. Amer. Math. Soc. **54**, 218–228 (1943); **63**, 595–598 (1948); in the latter paper the conjecture is proved for

quadratic irrationalities; MR **5**, 3; **10**, 34]. The author now shows that if $\theta \in C$ is of degree n and if $\theta > 2^n$, then $E(\theta)$ is a set of uniqueness (in particular, for each n there are at most a finite number of θ's for which the problem remains open). (3) The author establishes interesting connections between the properties of the sets of uniqueness and multiplicity on the one hand, and the properties of the ring W of continuous periodic functions having absolutely convergent Fourier series on the other. We state only one of his results. Consider the subset $I \subset W$ of functions vanishing in some domain (not necessarily always the same) containing the given closed set $E \subset (0, 2\pi)$. E is a set of uniqueness if and only if the weak closure (properly defined) of I coincides with W. *A. Zygmund* (Chicago, Ill.).

Satô, Masaka. Uniform convergence of Fourier series. Proc. Japan Acad. **30**, 528–531 (1954).

Let $f(x)$, of period 2π, be continuous with modulus of continuity $\omega(\delta)$, and let $\int_a^b f(x+t) \cos nt\, dt = O(1/\Phi(n))$ uniformly in x, a, b with $b-a \leqq 2\pi$, where $\Phi(n)$ increases with n. Also $\Theta(n)$ is given, where $\Theta(n)$ increases and $1 \leqq \Theta(n) \leqq \Phi(n)$. It is proved that there exist absolute constants A, B, and C such that

$$|s_n(x)-f(x)| \leqq \omega(1/n)[A \log \Theta(n) + B \log n/\Phi(n)] + C/\Theta(n),$$

where s_n denotes the nth partial sum of the Fourier series of f. [Cf. J. P. Nash, Rice Inst. Pamphlet. Special Issue, Nov. 1953, pp. 31–57; MR **15**, 619.] It follows that the Fourier series converges uniformly whenever

$$\Theta(n) \uparrow \infty, \quad \omega(1/n) \log \Theta(n) \to 0,$$

and

$$\omega(1/n) \log n/\Phi(n) \to 0.$$

W. W. Rogosinski (Newcastle-upon-Tyne).

James, R. D. Integrals and summable trigonometric series. Bull. Amer. Math. Soc. **61**, 1–15 (1955).

The author states the recent progress of the fundamental problem in the theory of trigonometrical series: to define the integral such that if a trigonometrical series

$$(*) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges everywhere to a function $f(x)$, then $f(x)$ is necessarily integrable and the coefficients, a_n and b_n, are given in the usual Fourier form. This problem was solved first by A. Denjoy [Ann. Sci. Ecole Norm. Sup. (3) **33**, 127–222 (1916); **34**, 181–236 (1917); C. R. Acad. Sci. Paris **172**, 653–655, 833–835, 903–906, 1218–1221; **173**, 127–129 (1921); Leçons sur le calcul des coefficients d'une série trigonométrique, t. I–IV, Gauthier-Villars, Paris, 1941, 1949; MR **8**, 260; **11**, 99].

The author begins by illustrating Denjoy's "totalisation of generalized second derivative," which is the inverse of the generalized second derivative; he defines it by the constructive method which consists of nine fundamental operations and is intrinsic as he says, even if much too complicated. The sum function $f(x)$ of the series (*) is integrable in the above sense and a_n and b_n are given by the integral of $f(x) \cos nx$ and $f(x) \sin nx$ with a suitable constant factor.

The second solution of the above problem was given by S. Verblunsky [Fund. Math. **23**, 193–236 (1934)], whose integral is the "approximate totalisation," which is the inverse of the approximate derivative. The third is the Marcinkiewicz-Zygmund integral [ibid. **26**, 1–43 (1936)], which is the inverse of the Borel derivative and is defined by the Perron method (or descriptive method). The fourth is the Burkill integrals: Cesàro-Perron integral and symmetric Cesàro-Perron integral [Burkill, J. London. Math. Soc. **11**, 43–48 (1936); Proc. London. Math. Soc. (3) **1**, 46–57 (1951); MR **13**, 126], which is the inverse of the Cesàro derivative.

Finally the author states the integrals due to W. H. Gage and himself [James and Gage, Trans. Roy. Soc. Canada. Sect. III. **40**, 25–35 (1946); MR **9**, 19; James, Canad. J. Math. **2**, 297–306 (1950); Trans. Amer. Math. Soc. **76**, 149–176 (1954); MR **12**, 94; **15**, 611]. The first integral is the Perron second integral which is almost the descriptive definition of the above Denjoy integral. By this integral the author gives a simple exposition of the Denjoy theory. The author gives further "the P^n-integral," the inverse of the generalized nth derivative, and shows that, if the series (*) is summable (C, k) to $f(x)$, then $f(x)$ is p^{k+2}-integrable and a_n and b_n are given as the P^{k+2}-integral of $f(x) \cos nx$ and $f(x) \sin nx$ with suitable constant factor. *S. Izumi* (Tokyo).

Putnam, C. R. A note on the Patterson functions. Quart. Appl. Math. **13**, 105–106 (1955).

Let $\rho(x)$ be a positive periodic function of period L; the corresponding Patterson function is

$$P(x) = L^{-1} \int_0^L \rho(t)\rho(t+x)dt.$$

If $\rho(x)$ has N maxima on $0 \leqq x < L$, crystallographers believe that $P(x)$ has at most $N(N-1)$ maxima in $0 < x < L$. Hartman and Wintner [Phys. Rev. (2) **81**, 271–273 (1951); MR **12**, 495] refuted this in the case $N=2$. Here the author goes even farther by constructing (very simply) functions $\rho(x)$ with $N=1$ for which $P(x)$ has any prescribed finite number of maxima on $(0, L)$. *R. P. Boas, Jr.*

Mautner, F. I. Note on the Fourier inversion formula on groups. Trans. Amer. Math. Soc. **78**, 371–384 (1955).

Let G be a separable, unimodular, locally compact group. Let $R = \int_Y R^y d\mu(y)$ be the canonical decomposition of the regular representation of R into factor representations. If $f \in L^1(G) \cap L^2(G)$, then $\int_G f(x) R_x^y dx$ (where R^y is the representation $x \to R_x^y$ and dx denotes Haar measure in G) defines (up to μ null sets) an operator-valued function $y \to F(y)$ on Y called the generalized Fourier transform of f. It is known from the work of Segal [Ann. of Math. (2) **52**, 272–292 (1950); MR **12**, 157] and the author of the paper under review [ibid. **52**, 528–556 (1950); MR **12**, 157] that the following Plancherel formula holds for the mapping $f \to F$: $\int_G |f(x)|^2 dx = \int_Y t_y[F(y)F(y)^*]d\mu(y)$, where t_y is a suitable normalization of the trace function for the factor W^y generated by the representation R^y. This normalization is of course independent of f. The principal result of the paper under review is that $f \to F$ may be extended to be a unitary map from a dense subspace of $L^2(G)$ onto the possibly incomplete Hilbert space of all operator-valued functions A, $y \to A(y)$, from y to the W^y which are measurable in a certain sense and such that $\|A\|^2 \int_Y t_y(A(y)A(y)^*)d\mu(y)$ exists and is finite. The question as to whether the class of operator-valued functions just described is complete or not is not discussed. In any event the generalized Fourier transform can be extended to be a unitary map of $L^2(G)$ onto the completion of the class.

In addition the paper contains a refinement of Theorem 1.1 of the author's paper cited above and some results on the inversion of $f \rightarrow F$. For example, if $g \in L^2(G)$ and $f = g * g^*$ is also in $L^2(G)$, then for all x in G, $f(x) = \int_Y t_y[(R_x^y)^* F(y)] d\mu(y)$ and the integral is absolutely convergent.

G. W. Mackey (Cambridge, Mass.).

***Harkevič, A. A. Spektry i analiz. [Spectra and analysis.]** 2d. ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 215 pp. 6.20 rubles.

Heuristic treatment of Fourier series and integrals as applied to engineering problems.

Eringen, A. Cemal. The finite Sturm-Liouville transform. Quart. J. Math., Oxford Ser. (2) **5**, 120–129 (1954).

This is a generalization of results given by E. J. Scott [same J. (2) **4**, 36–40 (1953); MR **14**, 869]. Let $\{\psi_n(y)\}$ be the normal orthogonal Sturm-Liouville system derived from the equation $L(u) = \lambda u$ $(L = q(x) - d^2/dx^2)$ with boundary conditions at $x = a$ and $x = b$ and $q(x)$ continuous in (a, b). The finite Sturm-Liouville (S.L.) transform of $f(x)$, continuous and integrable in (a, b), is defined to be the sequence $f_n = S(f) = \int_a^b f(y)\psi_n(y) dy$. When f is a function of two variables x and t, it is shown that the finite S.L. transform $S_x(G)$ of $G = q(x)f - \partial^2 f/\partial x^2$ depends on t and $f(a, t)$, $f(b, t)$, $f_x(a, t)$, and $f_x(b, t)$.

If, in a partial differential equation, the terms involving derivatives with respect to one of the variables can be reduced to the form G, the finite S.L. transform applied to this reduced form yields a sequence of ordinary differential equations. When these are solved, the solution of the original equation can be obtained by using the inversion theorem for S.L. normal orthogonal systems.

This method is applied to certain types of parabolic and elliptic equations and to the problem of heat conduction in a conical shell. *F. Goodspeed* (Stanford, Calif.).

Mayer-Kalkschmidt, Jörg. Zur Theorie der Laplace-Stieltjes-Integrale. Mitt. Math. Sem. Giessen no. **47**, ii+26 pp. (1954).

A well known theorem due essentially to Landau states that if $s(t)$ is monotone, and if the Laplace-Stieltjes transform: (*) $f(s) = \int_0^\infty e^{-st} ds(t)$ possesses a finite abscissa of convergence β_0, then $s = \beta_0$ is a singularity of $f(s)$. Now, (*) is said to be summable (C, k), $k \geqq 0$, if $\lim \sigma_k(s, t)$ for $t \rightarrow \infty$ exists where: $\sigma_k(s, t) = t^{-k} \int_0^t (t-\tau)^k e^{-s\tau} ds(\tau)$.

The author tries to derive theorems analogous to Landau's theorem for (C, k) summability. Thus, it is stated: Let $f(s)$ be given by (*) and let n be a positive integer. Denote the abscissa of (C, n) summability by β_n. If (i) $f(s)$ possesses a non-negative right-derivative at $s = \beta_n$, and (ii) $\sigma_n(\beta_n, t)$ increases monotonically for $t \geqq T \geqq 0$, then $s = \beta_n$ is a singular point for $f(s)$.

Unfortunately, this theorem is false for $n > 1$. Indeed, as was observed by the author for $T = 0$ (and is true also for $T > 0$), if there exists a (continuous) right-derivative at $s = \beta_n$, it follows from assumption (ii) that $f'(\beta_n) \leqq 0$. Thus, the theorem has meaning only if $f'(\beta_n) = 0$. In this case, however, it is wrong as the following example shows. Let:

$$f(s) = \int_{\pi/2}^{\infty} e^{-st} \frac{\sin t}{t} dt.$$

Then, $f(s) = (2i)^{-1} \log((s+i)(s-i)^{-1}) +$ entire function. Also, as an easy computation shows, $f'(0) = 0$ and

$$d\sigma_2(0, t)/dt = 2(1 - \sin t)/t^3 \geqq 0$$

for $t \geqq \frac{1}{2}\pi$, and $= 0$ for $0 \leqq t \leqq \frac{1}{2}\pi$, so that the conditions of the theorem hold, but obviously $s = 0$ is not a singularity.

Other curious mistakes were also noted. Among the correct results we mention the following: Let n be a positive integer, and $\beta_n > 0$ the abcissa of (C, n) summability. If $\sigma_n(0, t)$ increases monotonically for $t \geqq T \geqq 0$, then $s = \beta_n$ is a singular point of $f(s)$. *S. Agmon* (Jerusalem).

Erdélyi, A. On a generalisation of the Laplace transformation. Proc. Edinburgh Math. Soc. (2) **10**, 53–55 (1954).

Let

$$k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\eta + \frac{1}{2} + it)\Gamma(\rho + \frac{1}{2} + it)}{\Gamma(\eta + \alpha + \frac{1}{2} + it)} x^{-\frac{1}{2} - it} dt,$$

where we assume that neither Re $(\eta - \frac{1}{2})$ nor Re $(\rho - \frac{1}{2})$ is a negative integer, and consider the integral transformation $g(x) = \int_0^\infty k(xy) f(y) dy$. The author remarks that such an integral transformation is equivalent after a fractional integration by parts to a Laplace transform, that is $g(x) = \int_0^\infty e^{-xy}(xy^\rho I^+_{\eta,\alpha} f(y)\, dy$, where

$$I^+_{\eta,\alpha} f(x) = \{\Gamma(\alpha)\}^{-1} x^{-\eta-\alpha} \int_0^x (x-u)^{\alpha-1} u^\eta f(u)\, du.$$

Consequently, many known theorems concerning Laplace transforms can be transferred to the transform above. This work is related to the investigations of Varma [Proc. Edinburgh Math. Soc. (2) **8**, 126–127 (1949); MR **11**, 661] and Meijer [Nederl. Akad. Wetensch., Proc. **43**, 599–608 (1940); MR **2**, 96]. *I. I. Hirschman* (St. Louis, Mo.).

Arthurs, Edward, and Martin, Louis H. Closed expansion of the convolution integral (a generalization of servomechanism error coefficients). J. Appl. Phys. **26**, 58–60 (1955).

The integral $r(t) = \int_{-\infty}^{t} f(x) w(t-x)\, dx$ with $w(t)$ as the response function of a servomechanism and with $f(x)$ as forcing function is expanded by decomposition of $f(x)$. This function is split into a continuous part $f_c(x)$ and into a linear combination of unit step functions. It is assumed that the linear combination has a finite number of terms. The integral over f_c undergoes an integration by parts and leads to a convolution integral for the derivative of f_c, if this derivative exists. From here on the procedure is iterated, and a finite number of iterations leads to what is claimed as a closed expansion of the original convolution integral. The method is illustrated by an example for a tracking system. Such terms as "the derivative of the continuous portion of $f(x)$" indicate that a rigorous foundation of the method is still desirable. *H. Bückner* (Schenectady, N. Y.).

Polynomials, Polynomial Approximations

Cowling, V. F., and Thron, W. J. Zero-free regions of polynomials. Amer. Math. Monthly **61**, 682–687 (1954).

The authors consider the polynomial

$$P(z) = a_0 + a_1 z^{\lambda_1} + \cdots + a_n z^{\lambda_n},$$

where the $a_q \neq 0$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. The principal theorem is that all zeros of $P(z)$ lie in the closed region

$$|z| \leqq \max \left[\frac{1 + r_{k-1}}{r_k} \left| \frac{a_{k-1}}{a_k} \right| \right]^{1/\alpha_k},$$

where $r_0 = 0$, $r_n = 1$, and the remaining r_m $(m = 1, \cdots, n-1)$ are arbitrary positive numbers (their statement of this

theorem contains an unfortunate typographical error; other typographical errors occur in the text). This theorem is a generalization of the known result obtained by setting the $r_m=1$. *W. Leighton* (Pittsburgh, Pa.).

Nassif, M., and Makar, Ragy H. On non-algebraic basic sets of polynomials. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17**, 120–129 (1955).

The authors construct examples to show that in a number of theorems about algebraic basic sets of polynomials [particularly, Makar, Bull. Sci. Math. (2) **76**, 171–179 (1952); Duke Math. J. **21**, 75–78 (1954); Nederl. Akad. Wetensch. Proc. Ser. A. **57**, 57–68, 69–76 (1954); Makar and Makar, ibid. 306–318, 319–330 (1954); MR **14**, 547; **15**, 694, 955] the condition that the set is algebraic cannot be dropped. They also give two positive theorems on representation properties of square roots of simple monic sets whose coefficients satisfy certain inequalities. *R. P. Boas, Jr.*

Carlitz, L. Note on Legendre polynomials. Bull. Calcutta Math. Soc. **46**, 93–95 (1954).

The author proves the following theorem: Let $\{f_n(x)\}$ be a set of polynomials, degree $f_n(x)=n$, $f_0(x)=1$, $f_1(x)=x$, satisfying

$$(1-x^2)(f_n'^2-f'_{n-1}f'_{n+1})=n(n+1)(f_n^2-f_{n-1}f_{n+1})\quad (n\geqq 1).$$

Then $f_n(x)=P_n(x)$ for all $n=0, 1, 2, \cdots$. The fact that $P_n(x)$ satisfy this relation is not new [see Nanjundiah, Half-Yearly J. Mysore Univ. Sect. B. (N.S.) **11**, 57–61 (1950); MR **13**, 554] but this characterization presumably is. In addition, the author establishes explicit formulae for $dP_n^{(\lambda)}(\pm 1)/dx$, using only the generating function $(1-2xw+w^2)^{-\lambda}$ of the ultraspherical polynomial, thus generalizing a known result of Grosswald [Proc. Amer. Math. Soc. **1**, 553–554 (1950); MR **12**, 178] for $\lambda=1/2$. The proof is similar to du Plessis' for that case [ibid. **2**, 950 (1951); MR **13**, 553]. *A. B. Novikoff.*

Denisyuk, I. M. Some properties of polynomials analogous to Laguerre polynomials. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 79–81 (1954). (Ukrainian. Russian summary)

Denisyuk, I. M. Some integrals and expansions which contain normalized Laguerre polynomials and their analogues. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 165–167 (1954). (Ukrainian. Russian summary)

Denisyuk, I. M. Some integrals, matrices and approximations connected with polynomials analogous to the Laguerre polynomials. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 239–242 (1954). (Ukrainian. Russian summary)

In the first of these papers, polynomials $M_n(x)$ are defined by

$$(-1)^n-e^{-t}M_n(2t)=\int_0^t e^{-u}L_n(2u)du,$$

where $L_n(x)$ is the Laguerre polynomial. The generating function

$$\sum_{n=0}^{\infty} t^n M_n(x)=\frac{1}{1+t}\exp-\frac{xt}{1-t}$$

follows, and a differential equation of the second order, satisfied by M_n+M_{n+1} is obtained. In the second paper $\int_0^\infty e^{-x}L_m(x)M_n(x)\,dx$ is evaluated, and M_n is represented as a finite series of Laguerre polynomials. The third paper contains the evaluation of $\int_0^\infty e^{-x}M_m(x)M_n(x)\,dx$ and related formulas. *A. Erdélyi* (Pasadena, Calif.).

Moldovan, Elena. Observations sur certains procédés d'interpolation généralisés. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 477–482 (1954). (Romanian. Russian and French summaries)

For Fejér's interpolation polynomials [Math. Ann. **102**, 707–725 (1930)], which converge for every continuous function, the author shows that the order of the approximation is $\omega(n^{-1}\log n)$ if ω is the modulus of continuity of the function which is being approximated. *R. P. Boas, Jr.*

Freud, Géza. Über die Konvergenz des Hermite-Fejérschen Interpolationsverfahrens. Acta Math. Acad. Sci. Hungar. **5**, 109–128 (1954). (Russian summary)

For $n=1, 2, \cdots$ let $x_{1n}, x_{2n}, \cdots, x_{nn}$ be a set of n points on $(-1, 1)$ and let d_{kn}, $1\leqq k\leqq n$, be given constants. The Hermite-Fejér method of interpolation associates with a given function f on $(-1, 1)$ the sequence of polynomials $H_n(f, x)$ determined by the condition that $H_n(f, x)$ is of degree $\leqq 2n-1$ and agrees at x_{kn}, $k=1, 2, \cdots, n$, with f and has there the derivative d_{kn}. The author considers a positive weight-function $w(x)$ on $(-1, 1)$ and the corresponding sequence $p_n(x)$ of orthogonal polynomials, and takes the roots of p_n to be the points $x_{1n}, \cdots, x_{nn}$. He assumes that the p_n are uniformly bounded in an interior subinterval (α, β) of $(-1, 1)$ and that

$$d_{kn}=\begin{cases} o(n/\log n), & \alpha\leqq x_{kn}\leqq\beta \\ o(\min n(1-x^2_{kn})^{-1/2}, n^2), & \text{all } x_{kn}. \end{cases}$$

The following two theorems are proved in the paper. Theorem 1: Let the weight-function $w(x)$ be continuous in the closed interval $(-1, 1)$ and let w satisfy uniformly in (α, β) the condition

$$w(x_1)-w(x_2)=o\left(\log^{-1}\frac{1}{|x_2-x_1|}\right).$$

Then the interpolation sequence $H_n(f, x)$ converges at ξ, $\alpha\leqq\xi<\beta$, to $f(\xi)$, provided that f is bounded in $(-1, 1)$ and continuous at the points $\xi, -1, 1$. If f is continuous everywhere in (α, β), then the convergence is uniform in interior subintervals of (α, β). Theorem 2: Let $0<m\leqq w(x)\leqq M$ in $-1\leqq x\leqq 1$. Let f be continuous in $-1\leqq x\leqq 1$ and satisfy uniformly in (α, β)

$$f(x_2)-f(x_1)=o\left(\log^{-1}\frac{1}{|x_1-x_2|}\right).$$

Then the sequence $H_n(f, x)$ converges uniformly in each interior subinterval of (α, β) towards $f(x)$. *J. Wermer.*

Freud, Géza. On a theorem of Paul Erdös and Paul Turán. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 209–217 (1954). (Hungarian)

Hungarian version of a previous paper of the author [Acta Math. Acad. Sci. Hungar. **4**, 255–266 (1953); MR **15**, 620].

Fekete, Michel. Approximations par polynomes avec conditions diophantiennes. C. R. Acad. Sci. Paris **239**, 1337–1339 (1954).

Fekete, Michel. Approximation par des polynomes avec conditions diophantiennes. II. C. R. Acad. Sci. Paris **239**, 1455–1457 (1954).

Soit E un ensemble borné et fermé de points z dans le plan complexe, et $T(E)$ son diamètre transfini. Soit K un corps

quadratique imaginaire. Par noyau de E_1 relatif à K on désigne le sous-ensemble N de E formé par les racines de chaque equation $z^m+a_1 z^{m-1}+\cdots+a_m=0$, où les coefficients a_j sont des entiers de K, irréductible dans K, et dont les racines appartiennent toutes à E. L'auteur montre d'abord que si $T(E)<1$, alors N est vide, ou bien contient un nombre fini de points de E. Il démontre ensuite que les conditions nécessaires et suffisantes pour qu'une fonction $f(z)$ continu dans E admette des polynomes d'approximation uniforme dont les coefficients soient des entiers de K sont 1°) que $T(E)<1$ et 2°) que $f(z)$ prenne aux points z_ν $(\nu=1, 2, \cdots, n)$ du noyau des valeurs $f(z_\nu)$ telles que le polynome d'interpolation de Lagrange de f relatif aux points z_ν n'ait comme coefficients que des entiers de K. *R. Salem* (Paris).

Ferrer, Lorenzo. On the development of certain generating functions in a series of polynomials and their connections with the theory of the prepotential. Collect. Math. 6, 221–291 (1953). (Spanish)

Humbert has studied the partial differential equation of the third order in three independent variables x_1, x_2, x_3 which is obtained from $\partial^3 V/\partial u_1\partial u_2\partial u_3=0$ by setting

(1) $u_1=x_1+x_2+x_3,\quad u_2=x_1+jx_2+j^2x_3,\quad u_3=x_1+j^2x_2+jx_3,$

where j is a complex cube root of unity. In some sense Humbert's equation, $\Delta_3 V=0$, may be regarded as a generalization of Laplace's equation. The author replaces (1) by a linear transformation $u_r=\sum_s a_{rs}x_s$, and determines this transformation so as to make the resulting partial differential equation symmetric in x_1, x_2, x_3. In the (x_1, x_2, x_3)-space, a metric may be introduced which is determined by a cubic, rather than quadratic, differential form. The expansion of a suitable power of the distance of two points in this metric generates certain polynomials which may be regarded as analogous to Legendre polynomials. A suitable notion of orthogonality may be introduced, and the author studies the transformation of $\Delta_3 V$ under an orthogonal change of the independent variables.

Similar investigations are carried out also in 4-dimensional space.

The memoir contains a great many detailed results, and this review can attempt no more than an indication of the direction of the author's research. *A. Erdélyi.*

Special Functions

Kazarinoff, Nicholas D. Asymptotic expansions for the Whittaker functions of large complex order m. Trans. Amer. Math. Soc. 78, 305–328 (1955).

The author investigates the asymptotic behavior of the solutions of Whittaker's equation

$$\frac{d^2W}{dx^2}+\left\{-\frac{1}{4}+\frac{k}{x}+\frac{\frac{1}{4}-m^2}{x^2}\right\}W=0$$

for complex k, m, x when k is bounded, $|x|$ is arbitrary, and $m\to\infty$, it being assumed that

$$-\pi<\arg x\leqq\pi,\quad -3\pi/2<\arg m\leqq 3\pi/2.$$

This differential equation has a transition point at $x=2mi$. The author first obtains asymptotic forms valid, roughly speaking, when $(4m^2+x^2)^{1/2}$ is large, and then briefly indicates the asymptotic forms valid in the neighborhood of the transition point, where $(4m^2+x^2)^{1/2}$ is small.

The asymptotic forms are obtained by the method developed by R. E. Langer [same Trans. 34, 447–480 (1932); 67, 461–490 (1949); MR 11, 438; and other papers]. This method is extended, in the second part of the paper under review, to unbounded domains. [Reviewer's remark. Asymptotic solutions in unbounded domains have also been obtained by T. M. Cherry, ibid. 68, 224–257 (1950); MR 11, 596; and F. W. J. Olver in the paper reviewed below.] *A. Erdélyi* (Pasadena, Calif.).

Olver, F. W. J. The asymptotic solution of linear differential equations of the second order for large values of a parameter. Philos. Trans. Roy. Soc. London. Ser. A. 247, 307–327 (1954).

Asymptotic solutions of the differential equations

(1) $d^2w/dz^2=\{uz^n+f(z)\}w \quad (n=0, 1)$

are considered for large positive values of u. It is shown that solutions exist whose asymptotic expansions are given by series of the form

$$P(\xi)\left\{1+\sum_1^\infty A_s(z)/u^s\right\}+P'(\xi)\sum_0^\infty B_s(z)/u^{s+1},$$

where $P(\xi)$ is an exponential or Airy function according as $n=0$ or 1. The results are easily extended to complex u. The method of proof differs from those of earlier writers.

Equations (1) represent normalized forms of the equation

(2) $d^2w/dz^2=\{up(z)+q(z)\}w$

in which the variable z is confined to a simply-connected domain D of the complex plane. Three cases are considered: (a) $p(z)$ analytic with no zeros in D; (b) $p(z)$ analytic with one simple zero in D; (c) $p(z)$ analytic except for a pole of second order in D. The function $q(z)$ is assumed analytic except that in case (c) it may have a simple or double pole at the singularity of $p(z)$. The asymptotic series obtained in each of these three cases pertain to equation (2) in the normalized forms (1). This, of course, considerably simplifies the coefficients in these series.

Case (a) is classical, and the results of the paper offer no essential improvement over those of Birkhoff [Trans. Amer. Math. Soc. 9, 219–231 (1908)], Turrittin [Amer. J. Math. 58, 364–376 (1936)], and Langer [Trans. Amer. Math. Soc. 67, 461–490 (1949); MR 11, 438]. The present author considers complex z in a region which may be unbounded while previous discussions were for a real variable but for more general equations. The reviewer [see the paper reviewed above] has indicated the generalization of Langer's 1949 results to complex z in an unbounded region both for this case and case (b). It is emphasized that there exist solutions of (1) with the asymptotic expansions stated and which have as asymptotic forms partial sums of these series, i.e., that the solutions represented are independent of the number of terms taken in the partial sums. Others have perhaps not stated this clearly.

The discussion of case (c) is made part of the previous case. In this instance equation (2) may also be transformed into one with $n=0$, the singular point being at infinity. A neighborhood of infinity is allowed in the region for which (1) is discussed if $f(z)=O(|z|^{-2})$ as $|z|\to\infty$ uniformly with respect to $\arg z$. Cashwell [Pacific J. Math. 1, 337–352 (1951); MR 13, 461] has given the leading terms of the asymptotic series for the solutions of an equation more general than (2) under similar hypotheses. The reviewer and McKelvey [Bull. Amer. Math. Soc. 60, 369–370 (1954)]

have derived complete asymptotic series for the solutions of an equation similar to Cashwell's.

The discussion of case (b) improves upon results of Langer (1949) and Cherry [Trans. Amer. Math. Soc. **68**, 224–257 (1950); MR **11**, 596]. It is not stipulated that the domain of uniform validity of the expansions be bounded. Certain other restrictions, too involved to state here, are also removed. The author's proof gives a new scheme for estimating the error involved in approximating the solutions by their asymptotic forms. In a companion paper reviewed below application of these results is made to the asymptotic theory of the Bessel functions. *N. D. Kazarinoff.*

Olver, F. W. J. The asymptotic expansion of Bessel functions of large order. Philos. Trans. Roy. Soc. London. Ser. A. **247**, 328–368 (1954).

This paper contributes some interesting results to the asymptotic theory of Bessel functions. The contents are as follows: the expansion of $I_\nu(\nu z)$, $K_\nu(\nu z)$, $J_\nu(\nu z)$, $Y_\nu(\nu z)$, and the Hankel functions for large positive and also large complex orders; expansions for the derivatives; zeros of $J_\nu(z)$, $Y_n(z)$, $J_\nu'(z)$, and $Y_n'(z)$; zeros of the Hankel functions and their derivatives; tables for the calculation of zeros and associated quantities; some properties of Airy functions in the complex plane (n is a real positive parameter, ν may be complex). Similar expressions for these functions but not for their derivatives and zeros have been given by Cherry [Trans. Amer. Math. Soc. **68**, 224–257 (1950); MR **11**, 596]. The author's results are found by application of the theory of the paper reviewed above.

The expansions given for $I_\nu(\nu z)$ and $K_\nu(\nu z)$ are for $|\arg \nu|<\pi/2$ and $|\arg z|<\pi/2$ and hence are not useful in immediate neighborhoods of the turning points $z=\pm i$. The coefficients are computed to terms of order

$$\nu^{-9/2}(1+z^2)^{-1/4}\exp(\pm\nu\zeta),$$

where $\zeta=(1+z^2)^{1/2}+\ln\{z[1+(1+z^2)^{1/2}]^{-1}\}$. As the author notes, these expansions may also be obtained from the Debye expansions for $J_\nu(\nu z)$ and $H_\nu^{(1)}(\nu z)$ [Watson, Theory of Bessel functions, Cambridge, 2nd ed., 1944, pp. 262–268; MR **6**, 64]. The expansions for the functions $J_\nu(\nu z)$, $Y_\nu(\nu z)$, $H_\nu^{(1)}(\nu z)$, $H_\nu^{(2)}(\nu z)$, and their derivatives are given in terms of Airy functions. They are uniformly valid with respect to z (including the turning points) when $|\nu|$ is large. Recursion formulas for the coefficients are given, together with a procedure whereby explicit expressions for the higher coefficients may be obtained. The regions of validity of the various expansions are described. The author reverts the asymptotic series for the Bessel functions to obtain such series for their zeros. Both the distribution of and the values of the zeros are discussed. An example of the nature of the distribution of complex zeros is the following. If n is a positive integer, then, for $|\arg z|\leqq\pi$, $Y_n(nz)$ has, in addition to its infinite set of real positive zeros, two infinite strings of zeros asymptotically near to the negative axis of reals, together with $2n$ zeros asymptotically near to the boundary of an eye-shaped region in the z-plane whose extreme points are $z=\pm1$. Zeros of $Y_n(z)$ and $Y_n'(z)$ for general phase ranges of z, when n is integral, are discussed. Special attention is given to the case n half an odd integer.

The expansion of a zero of fixed enumeration is of the form

$$\nu p_0(\zeta)+\nu^{-1}p_1(\zeta)+\cdots,$$

where $\zeta=\nu^{-2/3}a$ and a is a corresponding Airy function zero. The coefficients $p_j(\zeta)$ are transcendental functions which may be pretabulated. It is remarked that "the expansion is uniformly valid with respect to all the zeros and numerically it is very powerful." However, the author does not make clear how the estimates upon the errors in the approximations for the zeros are obtained when he makes statements as to the number of significant figures in his results. Inasmuch as the series used are of an asymptotic nature this is an important consideration. It is stated, for example, that the series will give $j_{n,s}$ and $J_n'(j_{n,s})$ correct to at least ten significant figures when $n\geqq4$. It is intended to publish tables of Bessel function zeros and associated quantities having this same degree of accuracy.

The paper concludes with material on the asymptotic expansions of the Airy function $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ for complex z and on the distribution of and asymptotic expansions for their zeros. *N. D. Kazarinoff* (Lafayette, Ind.).

Schöbe, Waldemar. Eine an die Nicholsonformel anschliessende asymptotische Entwicklung für Zylinderfunktionen. Acta Math. **92**, 265–307 (1954).

Let $Z_\nu=\alpha J_\nu(z)+\beta Y_\nu(z)$, where α and β are complex constants. Let $F_{1/3}(y)$ and $F_{-2/3}(y)$ represent certain linear combinations of cylinder functions of orders 1/3 and $-2/3$, respectively, and which depend upon the choice of $Z_\nu(z)$. Then,

$$Z_\nu(z)\sim\left(\frac{2\gamma}{3z}\right)^{1/2}\sum_{\mu=0}^{\infty}(2\gamma z)^{-\mu/2}[S_\mu(y)F_{1/3}(y)+T_\mu(y)F_{-2/3}(y)],$$

where $\gamma=z-\nu$, $y=(2\gamma)^{3/2}/(3z^{1/2})$, and $S_\mu(y)$, $T_\mu(y)$ are polynomials in y. $S_\mu(y)$ and $T_\mu(y)$ are computed explicitly for $\mu\leqq4$. Similar series for $Z_\nu'(z)$ are developed. The author also presents his results in terms of Airy functions. The method of derivation of the results is analogous to that used in the derivation of the Debye series for $H_{z-\gamma}^{(2)}(z)$. It is accomplished by the use of Watson's lemma after rewriting the integrand in the Sommerfeld integral representation in suitable form.

Considerable attention is also paid to the establishment of asymptotic series for the zeros of $Z_\nu(z)$. In particular, it is shown that for the zero $\mathfrak{z}_{\nu,n}$

$$\mathfrak{z}_{\nu,n}\sim\nu-\sum_{k=0}^{\infty}s_k(a_n)(\nu/2)^{(1-2k)/3}\,|\arg\nu|\leqq\pi-\epsilon,$$

where a_n is the nth zero of the Airy function $\alpha\,\mathrm{Ai}(\zeta)-\beta\,\mathrm{Bi}(\zeta)$, and $s_n(a)$ is a polynomial in a. The s_n are computed for $n\leqq4$. A brief set of tables is included, e.g., of $3^{1/2}J_\nu(z)\pm Y_\nu(z)$ for $\nu=30$, $20\leqq z\leqq40$.

The results of this paper closely parallel and often overlap those recently obtained by F. Olver in the paper reviewed above. The methods of proof are distinct.

N. D. Kazarinoff (Lafayette, Ind.).

Singer, C. P. A new expansion for the modified Bessel function $K_0(z)$. Quart. J. Math., Oxford Ser. (2) **5**, 301–303 (1954).

Proof of

$$K_0(z)=-\left(\gamma+\log\frac{z}{2}\right)I_0(z)+2\sum_{n=1}^{\infty}\frac{1}{n}I_{2n}(z)$$

and related identities. *A. Erdélyi* (Pasadena, Calif.).

Ragab, F. M. New integrals involving Bessel-functions. Math. Z. **61**, 386–390 (1955).

"The integrals are

$$\int_0^\infty\lambda^{k-1}K_n(x\lambda^s)K_m(e^{i\pi}x\lambda^s)f(\lambda)\,d\lambda,$$

where $\epsilon=\pm1$ and where $f(\lambda)$ is one of the functions

$$e^{-\lambda},\quad J_l(2\lambda),\quad K_l(2\lambda),\quad e^{-\lambda}I_l(\lambda)."$$

(From the author's introduction.) *A. Erdélyi.*

Cioránescu, Nicolae. Une généralisation des fonctions de Bessel et leurs applications à l'intégration de certaines équations linéaires aux dérivées partielles d'ordre quelconque. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 499–509 (1954). (Romanian. Russian and French summaries)

The generalization in question is

$$\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma[n+(k+1)/m]z^{mn+k+p}}{m^p\Gamma[(k+1)/m]\Gamma(mn+k+1)\Gamma[n+p+(k+1)/m+1/2]},$$

and arises when the partial differential equation

$$\frac{\partial^m u}{\partial x^m}+\frac{\partial^m u}{\partial y^m}+\frac{m(p+1/2)}{y}\frac{\partial^{m-1}u}{\partial y^{m-1}}=0$$

is integrated by separation of the variables. *A. Erdélyi.*

Biedenharn, L. C., Gluckstern, R. L., Hull, M. H., Jr., and Breit, G. Coulomb functions for large charges and small velocities. Phys. Rev. (2) **97**, 542–554 (1955).

Coulomb wave functions are solutions of the differential equation $d^2y/d\rho^2+(1-2\eta\rho^{-1}-L(L+1)\rho^{-2})y=0$, and the authors wish to compute these functions for large values of the parameters η. They obtain expansions in powers of $\eta^{-1/3}$ of the values and derivatives of the standard Coulomb wave functions at $\rho=\rho_L=\eta+[\eta^2+L(L+1)]^{1/2}$ and at $\rho=\rho_0$; and expansions in terms of Bessel functions of variable $x=(\rho-\rho_L)[\rho_L^{-1}+L(L+1)\rho_L^{-3}]^{1/3}$ for ρ near ρ_L. They also use the JWKB method to find approximations for large η, and illustrate their results by numerical computations.

A. Erdélyi (Pasadena, Calif.).

Aoi, Tyûsei. On spheroidal functions. J. Phys. Soc. Japan **10**, 130–141 (1955).

The author is concerned with spheroidal wave functions of order zero, satisfying the differential equation

$$\frac{d}{dz}\left[(1-z^2)\frac{du}{dz}\right]+[\lambda+k^2z^2]u=0.$$

The methods and results obtained have been known for a long time from work by many authors, none of which is referred to. [Cf. J. Meixner and F. W. Schäfke, Mathieusche Funktionen und Sphäroidfunktionen, Springer, Berlin, 1954; MR **16**, 586.] *C. J. Bouwkamp* (Eindhoven).

Maximon, L. C., and Morgan, G. W. On the evaluation of indefinite integrals involving the special functions: development of method. Quart. Appl. Math. **13**, 79–83 (1955).

Maximon, L. C. On the evaluation of indefinite integrals involving the special functions: application of method. Quart. Appl. Math. **13**, 84–93 (1955).

"The solution of a general second-order linear partial differential equation in two variables by Laplace transforms is utilized to develop a method for the evaluation of a large class of indefinite integrals involving the special functions" (Authors' summary of the first paper). A simpler derivation of the principal formula is also given in the paper, and is attributed to P. Germain. Let $L=c_0+c_1D+c_2D^2$, $c_i=c_i(x)$, $D=d/dx$. Let $LY=0$, $Ly=f$, $W=Yy'-Yy'$. Then $c_2W'+c_1W=fY$, and

$$(*)\qquad \int\frac{fY}{c_2}\exp\left(\int\frac{c_1}{c_2}dx\right)dx=W\exp\int\frac{c_1}{c_2}dx.$$

Choosing Y, y, c_1, c_2, c_3 suitably, the authors derive two indefinite integrals involving confluent hypergeometric functions.

In the second paper the author uses (*) in combination with recurrence relations for the functions Y, y to derive a number of integrals involving Legendre functions, confluent hypergeometric functions, and Bessel functions. He also extends (*) to functions which satisfy a differential equation of order n. *A. Erdélyi* (Pasadena, Calif.).

Ross, Alan S. C. A note on two ${}_3F_2$'s. Math. Gaz. **39**, 61–63 (1955).

The author uses combinatorial analysis to obtain closed expressions for

$${}_3F_2\begin{bmatrix}1, & M, & m-M;\\ 1-M, & 1-m+M\end{bmatrix}\quad\text{and}\quad {}_3F_2\begin{bmatrix}1, & M+1, & m-M\\ 1-M, & 2-m+M\end{bmatrix},$$

where m, M are integers and $1\leq m\leq M$. The second series is well poised, and the author's result is a limiting case of Dixon's theorem [Erdélyi et al., Higher transcendental functions, vol. I, McGraw-Hill, New York, 1953, p. 189; MR **15**, 419]; the first series is nearly poised.

A. Erdélyi (Pasadena, Calif.).

***Emersleben, Otto. Über zwei Epsteinsche Zetafunktionen 4. und 8. Ordnung.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 233–250. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

Let

$${}^{(p)}Z\begin{vmatrix}0\\0\end{vmatrix}(s)=\sum_{n=1}^{\infty}r_p(n)n^{-s/2},\qquad {}^{(p)}Z\begin{vmatrix}0\\ \frac{1}{2}\end{vmatrix}(s)=\sum_{n=1}^{\infty}(-1)^n r_p(n)n^{-s/2},$$

where $r_p(n)$ is the number of representations of n as a sum of p squares of integers. For $p=4$, 8 the author proves the functional equations

$${}^{(p)}Z\begin{vmatrix}0\\0\end{vmatrix}(s)=c_p(s)\cdot{}^{(p)}Z\begin{vmatrix}0\\ \frac{1}{2}\end{vmatrix}(s),$$

where

$$(2^{1-s/2}+1)c_4(s)=2^{2-s/2}-1,\quad (2^{4-s}-2^{1-3/2}+1)c_8(s)=2^{4-s/2}-1.$$

He investigates these functions more closely when $p=4$, 8, giving values when $s=p$, and proving the existence of sets of equidistant zeros on the critical lines. *A. Erdélyi.*

Legendre, Robert. Les fonctions et intégrales elliptiques à module réel en mécanique des fluides. O.N.E.R.A. Publ. no. **71**, i+74 pp. (1954).

The main part of this paper is expository, giving a summary derivation of the properties of elliptic integrals, and of elliptic, theta and zeta functions. In the last 12 pages applications are given to 2-dimensional aerofoil profiles and to the vibrations of a thin swallow-tail wing.

L. M. Milne-Thomson (Greenwich).

Harmonic Functions, Potential Theory

Deny, Jacques. Familles fondamentales. Noyaux associés. Ann. Inst. Fourier Grenoble **3** (1951), 73–101 (1952).

In H. Cartan and J. Deny, Acta Sci. Math. Szeged **12**, Pars A, 81–100 (1950) [MR **12**, 257] a general potential theory was formulated for locally compact Abelian groups G. The author now modifies and generalizes the preceding theory and develops certain aspects of it in more detail. A fundamental family is defined as a set Σ of measures $\sigma \geqq 0$ to which can be associated a measure $\kappa \geqq 0$ (called the base of Σ) satisfying (1) $\kappa * \sigma$ is defined for all $\sigma \in \Sigma$ and $\kappa - \kappa * \sigma$ is $\geqq 0$, not identically 0, and of compact support, and (2) every neighborhood of the origin supports a corresponding $\kappa - \kappa * \sigma$. If, further, (3) $\kappa * \sigma^p \to 0$ vaguely as $p \to \infty$ for all $\sigma \in \Sigma$, then the base κ is called a kernel for Σ. It is proved here that a kernel always exists, uniquely to within a positive factor, for a given fundamental family (the previous paper assumed the existence of a kernel and required $\sigma(G) \leqq 1$). Motivation for the theory is supplied by the Newtonian potential in $G = R^n$ ($n \geqq 3$), in which case Σ is the family of uniform distribution σ of the mass $+1$ on spherical surfaces and κ is the measure having density $|x|^{2-n}$ (i.e. the kernel of the Newtonian potential).

A real measure U such that $U * \sigma$ is defined and $\geqq U$ for all $\sigma \in \Sigma$ is called subharmonic (harmonic if $U * \sigma = U$). For all constants to be harmonic it is necessary and sufficient that $\sigma(G) = 1$ for every $\sigma \in \Sigma$. If U is a subharmonic measure, then there exists a unique measure $\mu \leqq 0$ (called the measure associated with U) such that $(\epsilon - \sigma) * U = [(\epsilon - \sigma) * \kappa] * \mu$, where ϵ is the measure $+1$ at the origin. This leads to the following generalization of the Riesz decomposition theorem: if U is a subharmonic measure admitting a harmonic majorant, μ the measure associated with U, and κ a kernel for Σ, then $U - \kappa * \mu$ is harmonic. Further analogues of the properties of Newtonian potentials are established, and the sweeping out process is discussed in detail.

A number of examples are given, among which there is exhibited a harmonic measure which is not absolutely continuous with respect to the invariant measure of the group. The treatment of harmonic and subharmonic measures (rather than functions) is thus a significant extension of the classical theory. On the other hand, the fine structure of Newtonian potentials is lost, since the subharmonic measures correspond only to almost subharmonic functions. It should be remarked also that the logarithmic potential cannot be subsumed under this general theory, in view of the fact that the base is required to be $\geqq 0$.

M. G. Arsove (Seattle, Wash.).

Pini, Bruno. Su un integrale analogo al potenziale logaritmico. Boll. Un. Mat. Ital. (3) **9**, 244–250 (1954).

The author defines generalizations of the operator $M = \partial^2/\partial x^2 + \partial/\partial y$ analogous to the operators of Blaschke and Privaloff associated with $\partial^2/\partial x^2 + \partial^2/\partial y^2$. For a continuous function $u(x, y)$, consider the mean values

$$\mu_0(x, y; r) = (2\pi)^{-1/2} \int_{-\pi/2}^{\pi/2} u(\xi, \eta)(\cos\theta)(\log(1/\sin^2\theta))^{1/2} d\theta,$$

where $\xi = x + 2^{1/2} r (\sin\theta)(\log(1/\sin^2\theta))^{1/2}$, $\eta = y + r^2 \sin^2\theta$, and

$$\mu_1(x, y; r) = 2r^{-2} \int_0^r \mu_0(x, y; \rho)\rho\, d\rho.$$

Let $M_0 u(x, y)$ and $M_1 u(x, y)$ denote the limits, as $r \to 0$, of

$$3^{3/2}[\mu_0(x, y; r) - u(x, y)]/r^2$$

and

$$2(3^{3/2})[\mu_1(x, y; r) - u(x, y)]/r^2,$$

provided that these limits exist. When M_0 exists, then M_1 does and $M_0 = M_1$. If u is continuous and $\partial^2 u/\partial x^2$, $\partial u/\partial y$ exist, then $M_0 = M_1 = M$. The author considers boundary-value problems associated with the inhomogeneous equation $M_0 u = f$. *P. Hartman* (Baltimore, Md.).

Plume, Z. Ya. Solution of Neumann's problem in potential theory in the case of an open contour. Latvijas PSR Zinātņu Akad. Vēstis **1951**, no. 5 (46), 815–821 (1951). (Russian. Latvian summary)

In the plane w of $z = x + iy$ is given a smooth open arc $L = ab$, whose curvature is of Hölder class H ($H(\lambda)$ is the Hölder class, when the Hölder constant is λ). The positive direction along L is from a to b. $\Phi(z)$ is p.a. (piece-wise analytic), if it is analytic for z finite in $w - L$ and is continuously extendible to L from the left $(+)$ and from the right $(-)$, while $\Phi(z) = O(|z - c|^{-\alpha})$ ($\alpha < 1$) for z in $w - L$ near c, where c is a or b. If $\phi(t) \in H(\lambda)$ on every closed part of L, disjoint from a, b, and if $\phi(t) = \phi^*(t)(t - c)^{-\alpha}$, where $\phi(t) \in H$, near c, then $\phi(t) \in H^*$ on L. The Neumann problem for L consists in finding $u(x, y)$, harmonic in $w - L$, such that: (1) u is $\operatorname{Re}\Phi(z)$ (real part of Φ), where $\Phi(z)$ is p.a., (2) $\Phi(\infty) = 0$, (3) $|\Phi(z)|$ is bounded, (4) $du^+/dn = du^-/dn = f$ on L, f of class H^* being assigned. The solution is found in the form

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\phi(t)\, dt}{t - z} = u + iv;$$

the real function ϕ is determined with the aid of a singular integral equation (in the sense of principal values).

W. J. Trjitzinsky (Urbana, Ill.).

Komatu, Yûsaku. On transference of boundary value problems. Kōdai Math. Sem. Rep. **1954**, 71–80 (1954).

Komatu, Yûsaku. A supplement to "On transference of boundary value problems". Kōdai Math. Sem. Rep. **1954**, 97–100 (1954).

Let D be a domain in the xy-plane which is bounded by a smooth Jordan curve C, and let V be a measurable function on C whose mean value over C vanishes. The author studies the relations between the functions u and v which are harmonic in D and determined (in the case of v only up to an additive constant) by the boundary data $u = V$ and $\partial v/\partial n = V$, respectively. If D is a circle these relations are very simple indeed, but this simplicity does not carry over to other domains. The author gives a complete treatment of the cases in which D is the extended plane bounded by a rectilinear or circular slit, respectively. Special consideration is given to the end-points of the slits at which the Neumann problem is apt to cause trouble. *Z. Nehari*

Komatu, Yûsaku. Über eine Übertragung zwischen Randwertaufgaben für einen Kreisring. Kōdai Math. Sem. Rep. **1954**, 101–108 (1954).

Let u and v be two functions which are harmonic in the ring $\rho < |z| < 1$ and which are connected by the boundary relations $\partial v/\partial r = -u(r = 1)$ and $r\partial v/\partial r = u(r = \rho)$. The author shows that u and v are related to each other in a simple way and he uses this relation to transform Villat's formula into a formula solving the Neumann problem for the ring.

Z. Nehari (Pittsburgh, Pa.).

Komatu, Yûsaku, and Mizumoto, Hisao. On transference between boundary value problems for a sphere. Kōdai Math. Sem. Rep. **1954**, 115–120 (1954).

Let Δ be the n-dimensional Laplace operator and let u and v be solutions of $\Delta u = \Delta v = 0$ which are harmonic in the unit hypersphere U and satisfy the boundary relation $u = \partial v/\partial r$ (r being the distance from the origin). The authors prove, in what appears to be a needlessly elaborate fashion, that the relation (*) $u = r\partial v/\partial r$ holds identically in the interior of U. (*) is an immediate consequence of the easily verified fact that $r\partial v/\partial r$ is harmonic if the same is true of v.

Z. Nehari (Pittsburgh, Pa.).

Sibirani, Filippo. Sulla risoluzione del problema di Neumann in campi prossimi a quelli classici. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **9**, 3–6 (1952).

In three notes [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **25**, 1° semestre, 413–417, 499–502, 579–581 (1916)], U. Cisotti discussed the solution of the problems of Dirichlet and Neumann. The present author gives an example which shows that an assumption made by Cisotti concerning the deformation of a surface is not in general true.

E. T. Copson (St. Andrews).

Sretenskiĭ, L. N. On the uniqueness of determination of the form of an attracting body from the values of its external potential. Dokl. Akad. Nauk SSSR (N.S.) **99**, 21–22 (1954). (Russian)

Verf. fragt nach Bedingungen dafür, dass zwei mit Masse konstanter Dichte erfüllte Körper C_1 und C_2, die im Aussenraum dasselbe Newtonsche Potential haben, miteinander identisch sind. Hierzu dient folgende Festsetzung: Existiert eine Ebene P so, dass jede zu ihr senkrechte Gerade die Oberfläche eines Körpers C in nur zwei Punkten, die auf verschiedenen Seiten von P liegen, schneidet, so sagt man, C besitze eine mittlere Ebene. Nunmehr gilt folgender Satz: Besitzen zwei Körper C_1 und C_2 von gleicher konstanter Dichte je eine mittlere Ebene, die beide zueinander parallel sind, liegen die Schwerpunkte im Innern und sind die Potentiale im Äusseren gleich, so sind C_1 und C_2 identisch. Der Beweis, der zum Teil nur angedeutet ist, benutzt Untersuchungen von Novikoff [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **18**, 165–168 (1938)].

K. Maruhn.

Pini, Bruno. Sulle funzioni sub e super-biarmoniche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 702–707 (1954).

L'auteur rectifie un critère de biharmonicité de Nicolesco analogue au critère d'harmonicité de Blaschke, introduit les notions de sous- et sur-biharmonicité et en donne des critères analogues et quelques propriétés. Extension possible aux fonctions polyharmoniques.

M. Brelot (Paris).

Drăganu, Mircea. Une méthode pour résoudre le problème biharmonique fondamental pour le cercle, à l'aide des fonctions analytiques. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 517–525 (1953). (Romanian. Russian and French summaries)

The author finds an explicit expression for the function $U = U(x, y)$ which satisfies

$$\Delta\Delta U = 0,\quad x^2+y^2 \leqq 1,\quad \frac{\partial U}{\partial x} = f_1,\quad \frac{\partial U}{\partial y} = f_2,\quad x^2+y^2 = 1,$$

by expanding all functions in Fourier series. *A. Erdélyi.*

Differential Equations

Viktorovskiĭ, E. E. On a generalization of the concept of integral curves for a discontinuous field of directions. Mat. Sb. N.S. **34**(76), 213–248 (1954). (Russian)

Let $f(x, y)$ be measurable on a bounded region G of the xy-plane; let a summable function $M(x)$ exist such that $|f(x, y)| \leqq M(x)$ for almost all y. The function $u(x)$ is said to be a generalized solution of the differential equation $y' = f(x, y)$ through (x_0, y_0) if u is defined and absolutely continuous on an interval I: $[x_0, x_0+a]$, $u(x_0) = y_0$, $|u(x)-y_0| < \int_{x_0}^{x}[M(x)+k]dx$ for some $k>0$ and if, for every $\epsilon>0$ and every null set N in G, there exists $v(x)$, x in I, such that for each x one has $|u(x)-v(x)| < \epsilon$, $|v(x)-\int_{x_0}^{x} f(x, v(x))dx| < \epsilon$, and $(x, v(x))$ not in N. It is proved that such solutions exist and that there is a maximal such solution $u_0(x)$; furthermore, if $f(x, y)$ is continuous in y at $y = u_0(x)$ on a measurable subset K of I, then $u_0'(x) = f(x, u_0(x))$ almost everywhere in K. These results are generalized to n dimensions. It is shown that $u(x)$ is a generalized solution if and only if for each x at least one of the following conditions holds: (a) for every interval J: (y_1, y_2) containing $u(x)$ and every $\epsilon>0$ the set $\{y | y \text{ in } J;\ |u'(x)-f(x, y)| < \epsilon\}$ has positive measure; (b) for every J the subsets of J on which $f(x, y) > u'(x)$ and on which $f(x, y) < u'(x)$ have positive measure. Various results of Kneser, Kamke, Carathéodory and others on the structure of the family of solutions are extended to the generalized solutions. For example, it is shown that the points $(x, u(x))$ on all solutions through (x_0, y_0) form a continuum; this is also established for the n-dimensional case.

W. Kaplan (Ann Arbor, Mich.).

Petropavlovskaya, R. V. On the existence and uniqueness of solution of a system of differential equations of a certain class. Mat. Sb. N.S. **36**(78), 149–162 (1955). (Russian)

Consider the system

$$(1)\qquad \frac{dx_i}{dt} = f_i(t, x)\Lambda_i(t)/B_i(x) \qquad (x = (x_1, \cdots, x_n);\ i = 1, \cdots, n).$$

Here the f_i, Λ_i, and $B_i(x)$ are continuous in the region $D\{0 \leqq t \leqq a \leqq 1, |x_i| \leqq b\}$, for $0 < t \leqq a$ and $|x_i| \leqq b$; (1°) $0 < m \leqq |f_i| \leqq M$ in D; (2°) for some $N>0$ and $0 \leqq \alpha < 1$, if the segments $[x_i', x_i'']$ do not contain 0, one has

$$|f_i(t, x')-f_i(t, x'')| \leqq N\sum_1^n |x_j'-x_j''|\nu^{-1} \qquad (\nu = \min(|x_j'|^\alpha, |x_j''|^\alpha));$$

$$rt^{\gamma_i} \leqq \Lambda_i(t) \leqq Rt^{\gamma_i} \quad (r>0, R>0, \gamma_i > -1);$$

$$l|u|^{\beta_i} \leqq B_i(u) \leqq L|u|^{\beta_i} \quad (l>0, L>0; \beta_i \geqq 0).$$

Under the above conditions the following is proved. The system can be transformed into

$$(2)\qquad \frac{dz_i}{dt} = F_i(\tau, z)\tau^{\bar{\gamma}_i},$$

where $\bar{\gamma}_i \geqq 0$, the F_i are continuous and satisfy (1°), (2°) in some region D^*; (2) has a solution continuous for $0 \leqq \tau \leqq H$, $z_i(0) = 0$; there is a unique system $x_i(t)$, continuous for $0 \leqq t \leqq H$, satisfying (1) for $0 < t \leqq H$, $x_i(0) = 0$, such that $|x_i(t)| \leqq b$; it is stated how one is to choose H. In this work use is made of the method of successive approximations and it is indicated that similar results can be obtained under some more general conditions.

W. J. Trjitzinsky.

Urabe, Minoru, and Katsuma, Shôichirô. Generalization of Poincaré-Bendixson theorem. J. Sci. Hiroshima Univ. Ser. A. **17**, 365–370 (1954).

Let (E) $\dot{x}=X(x, y)$, $\dot{y}=Y(x, y)$, where X and Y are real $C^{(1)}$ functions on an open plane region G and let two simple closed curves C_1 and C_2 (of class $C^{(2)}$) bound a closed annular region $R\subset G$ which is free of critical points of the differential system. The authors improve the Poincaré-Bendixson theorem by establishing the existence of a limit cycle of (E) in R under the hypothesis that at each point of the boundary of R the vector field of (E) either points into (out of) R or else is tangent to the boundary curve of R. *L. Markus.*

Gomory, Ralph E. Trajectories tending to a critical point in 3-space. Ann. of Math. (2) **61**, 140–153 (1955).

Consider $dv/dt=F(v)$, where $v(t)$ is a vector in real n-space which for $t\to\infty$ approaches a critical point P, say the origin; thus the differentiable vector-valued function $F(v)$ vanishes at $v=0$. By an extension of the radius vector, one maps E^n-P on the exterior of the unit sphere D' in E^n. The author considers the set $L(v)$ (or $L(v')$ when considered on D') of positive limit directions of the solution $v(t)$, by means of an associated autonomous differential system on D'. If $n=3$ then $D'=S^2$ and the well-known structure of autonomous differential systems on S^2 can be interpreted to yield information about $L(v)$. Typical of the many results is the following: $L(v')$ either contains a critical point of the system on D' or else is a closed curve. *L. Markus.*

✶Friedrichs, K. O. Fundamentals of Poincaré's theory. Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 56–67. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

The author considers the problem of continuation of periodic solutions, from $\mu=0$, in the differential system $\dot{u}_i=f_i(u_1, \cdots, u_n, t; \mu)$, where the f_i have period T in t. For the degenerate case, i.e., the variational equation has solutions with period T, one may have no continuation for $\mu>0$ or a bifurcation. The author reduces the degenerate problem to a non-degenerate modified problem by introducing new parameters. Then the relation between the periodic solutions of the modified problem and the original problem is reduced to a study of "bifurcation equations." The author indicates that a detailed exposition of this theory is being planned for publication. *L. Markus.*

✶Kaplan, Wilfred. Some methods for analysis of the flow in phase space. Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 99–106. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

The author discusses generally techniques for finding periodic solutions and other important features of real ordinary differential systems $\dot{x}_i=f_i(x_1, x_2, \cdots, x_n)$ $(i=1, 2, \cdots, n)$ by considering these as the characteristic equations of a partial differential equation

$$\frac{\partial F}{\partial t}+\sum_{i=1}^{n} f_i(x_1, x_2, \cdots, x_n)\frac{\partial F}{\partial X_i}=0.$$

The concepts of classical mechanics and hydrodynamics are used to illuminate the relation between the ordinary system and the partial differential equation. *L. Markus.*

Kimura, Toshifusa. Sur les points singuliers des équations différentielles ordinaires du premier ordre. II. Comment. Math. Univ. St. Paul. **3**, 43–49 (1954).

[For part I see same Comment. **2**, 47–53 (1954); MR **15**, 705.] Studied is the equation (1) $xdy/dx=P(x, y)/Q(x, y)$, where P, Q are polynomials in y with coefficients analytic in x for $x=0$; $P_0(y)=P(0, y)\neq 0$, $Q_0(y)=Q(0, y)\neq 0$; $v=Q_0'(y)/P_0'(y)$ is $Q_0(y)/P_0(y)$ reduced. It is assumed (A) that there is no combination of the residues of v having an imaginary sum; it is proved that every solution of (1) tends to a zero of $P_0(y)$ when x $(>0)\to 0$. Without hypothesis (A), every solution of (1) tends to a zero of $P_0(y)$ when $x\to 0$ along the path $x=\exp(r\tau^{-1}e^{i\theta})$ $(-\infty<r\leqq -R$, τ being a suitable positive integer) for every $|\theta|<\frac{1}{2}\pi$, a finite number of values of θ excepted. *W. J. Trjitzinsky.*

Pliś, A. On a topological method for studying the behaviour of the integrals of ordinary differential equations. Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 415–418 (1955).

The present paper is a further contribution to Ważewski's topological theory of the asymptotic behavior of solutions of differential systems (*) $dx_i/dt=f_i(t, x_1, \cdots, x_n)$, $i=1, 2, \cdots, n$ [see T. Ważewski, Ann. Soc. Polon. Math. **20**, 279–313 (1948); MR **10**, 122; F. Albrecht, Bull. Acad. Polon. Sci. Cl. III. **2**, 315–318 (1954); MR **16**, 248, cf. this second review for notations]. The following modification of previous theorems of the named authors is proved. If $(t, x_1, \cdots, x_n)\in\Omega$, where Ω is an open set of E_{n+1}, if ω is an open subset of Ω, if S is the set of all points of egress (end points) of $\Omega B(\omega)$ [$B(\omega)$ boundary of ω], and these points are all strict points of egress, if Z_1, S_1 are certain sets with $S_1\subset S$, $Z\subset\omega+S_1$, $Z\neq 0$, such that S_1 is no homotopic retraction of Z, then there exists at least one point $P_0\in Z$ such that the solution of (*) having its first point in P_0 either remains indefinitely in ω, or ends up at a point of $S-S_1$.
L. Cesari (Lafayette, Ind.).

Latişeva, K. I. Normal solutions of linear differential equations with polynomial coefficients. Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **8**, no. 4(11), 116–124 (1954). (Romanian)

Translated from Uspehi Mat. Nauk (N.S.) **8**, no. 5(57), 205–212 (1953); MR **16**, 476.

Antosiewicz, H. A., and Abramowitz, Milton. A representation for solutions of analytic systems of linear differential equations. J. Washington Acad. Sci. **44**, 382–384 (1954).

"In this note we consider an n-dimensional differential equation (1) $\dot{x}=A(t, z)x$ where $A(t, z)$ is analytic in the scalar variable t and continuous in the k-dimensional vector parameter z for all $|t|\leqq t<\infty$ and $\|z\|\geqq\zeta\geqq 0$. . . . Let (3) $\dot{y}=B(t)y$ be another n-dimensional vector differential equation with $B(t)$ analytic in t for $|t|\leqq t<\infty$, . . . whose general solution $y(t)=Y(t)c$ is assumed to be known. . . . Theorem. The general solution of (1) may be represented in the form $x(t, z)=X(t, z)Y(t)c$, where $X(t, z)$ is analytic in t and continuous in z for $|t|<\sigma<\tau$ and $\|z\|\geqq\zeta\geqq 0$." In applications (3) may approximate (1) in an appropriate sense.
F. A. Ficken (Knoxville, Tenn.).

Smirnov, M. M. On the integration of a system of differential equations. Prikl. Mat. Meh. **19**, 127–128 (1955). (Russian)

The paper gives a solution of the system of two simultaneous differential equations for the transfer of heat in

cross-flow. Two steady streams flowing at right angles to each other (directions x and y, respectively) are separated by a plate and it is desired to determine the temperature distributions $\theta(x, y)$ and $T(x, y)$ in the fluids on both sides of the plate. The problem was formulated and solved by W. Nusselt [Z. Verein. Deutsch. Ingenieure **55**, 2021–2024 (1911); and Forschg. Ing.-Wes. **1**, 417 (1930); also E. Schmidt, Einführung in die technische Thermodynamik ..., 4th ed., Springer, Berlin, 1950, p. 379], but the paper under review gives a closed-form solution based on Erugin's general solutions [Uč. Zap. Leningrad. Gos. Univ. Ser. Mat. **16** (1949)].

The equations for $\theta(x, y)$ and $T(x, y)$ are

$$\frac{\partial\theta}{\partial x}=b(T-\theta) \quad \text{and} \quad \frac{\partial T}{\partial y}=-a(T-\theta),$$

and the boundary conditions are $T(x, 0)=1$ and $\theta(0, y)=0$, all quantities having been made non-dimensional. The solutions are:

$$T(x, y)=e^{-ay-bx}\left[b\int_0^x e^{bt}J_0(2i(aby(x-t))^{1/2})\,dt + J_0(2i(abxy)^{1/2})\right]$$

and

$$\theta(x, y)=-ie^{-ay-bx}\left[b^2\int_0^x e^{bt}J_1(2i(aby(x-t))^{1/2})\frac{(x-t)dt}{(aby(x-t))^{1/2}} + \frac{bxJ_1(2i(abxy)^{1/2})}{(abxy)^{1/2}}\right].$$

J. Kestin (Providence, R. I.).

Krasovskiĭ, N. N. On stability in the large of the solution of a nonlinear system of differential equations. Prikl. Mat. Meh. **18**, 735–737 (1954). (Russian)

Let the n-vector system (1) $\dot{x}=X(x)$ be such that $X(0)=0$ and that X is of class C^1 in the whole space. Let J denote the Jacobian matrix of X. Theorem: In order that the origin be asymptotically stable for the whole space it is sufficient that there exist a symmetric matrix A such that for all x the characteristic roots of $\frac{1}{2}(AJ+J'A)$ all lie below a certain fixed $-\delta$, $\delta>0$. [Additional reference: Erugin, Prikl. Mat. Meh. **15**, 227–236 (1951); MR **12**, 705.] *S. Lefschetz.*

Ku, Y. H. Analysis of nonlinear systems with more than one degree of freedom by means of space trajectories. J. Franklin Inst. **259**, 115–131 (1955).

The procedure employed is to represent the solution of nonlinear differential equations of higher order than the second in a multidimensional phase space. Actually, the author carries through the details only in the three-dimensional case, corresponding to third-order differential equations, some of which describe special electric circuits having two degrees of freedom. The two-dimensional phase-plane representation is valuable because it gives much information for a moderate amount of work. The author's three-dimensional graphs clearly cost substantial labor to prepare; it would be important to know more about the "economics" of the process to judge its usefulness. *E. Pinney.*

Taam, Choy-Tak. An extension of Osgood's oscillation theorem for a nonlinear differential equation. Proc. Amer. Math. Soc. **5**, 705–715 (1954).

The solutions of

$$\frac{d}{dt}\left(m(t)\frac{dx}{dt}\right)+\sum_{i=1}^{k}f_i(t)x^{2i-1}=0$$

are shown, under conditions too lengthy to state here, to be oscillatory with bounded amplitudes of oscillation. This extends a theorem of W. F. Osgood [Bull. Amer. Math. Soc. **25**, 216–221 (1919)] for the case $k=1$, $m(t)\equiv 1$. Another theorem on boundedness of solutions is proved, extending a result of W. Leighton [Proc. Nat. Acad. Sci. U. S. A. **35**, 190–191, 422 (1949); MR **10**, 708] for $k=1$.

G. E. H. Reuter (Manchester).

Suyama, Yukio. On the non-oscillatory solution of second-order linear differential equation. Mem. Fac. Sci. Kyūsyū Univ. A. **8**, 207–212 (1954).

The author tries to prove that the differential equation (*) $y''(x)+q(x)y(x)=0$ is non-oscillatory in $0\leqq x<\infty$ if $q(x)$ is continuous in $0\leqq x<\infty$ and $L=\lim_{T\to\infty} T^{-1}\int_0^T(\int_0^t q(s)ds)dt$ exists and is finite. Unfortunately, the theorem is false, as shown by the oscillatory equation $y''(x)+\alpha(x+1)^{-2}y(x)=0$ $(\alpha>1/4)$, for which $L=\alpha$. Incidentally, this theorem would imply the even more striking, yet equally false, assertion that (*) is non-oscillatory if $q(x)$ is summable in $(0, \infty)$.

Z. Nehari (Pittsburgh, Pa.).

Atkinson, F. V. On linear perturbation of non-linear differential equations. Canad. J. Math. **6**, 561–571 (1954).

Solutions of $y''+y^{2n-1}+g(x)y=0$ $(n\geqq 2$, $g(x)$ small for large $x)$ are classified according to the behaviour of their amplitude $r(x)$, defined by $r^{2n}=y^{2n}+n(y')^2$. Under mild restrictions on $g(x)$, $r(\infty)$ always exists. Solutions for which $r(\infty)>0$ are of type I; this includes all solutions for which $|y(0)|+|y'(0)|$ is large enough. Solutions for which $r(\infty)=0$ are of type II; they do not occur, roughly, unless $g(x)$ is negative and not too small. Further results relate to non-oscillation of type II solutions, bounds for the amplitude, oscillation of type I solutions, and (for suitably restricted $g(x)$) an asymptotic formula for type I solutions in terms of solutions of the "unperturbed" equation $y''+y^{2n-1}=0$. The methods of proof are mainly based on those for the well-known case $r=1$. *G. E. H. Reuter.*

Atkinson, F. V. The asymptotic solution of second-order differential equations. Ann. Mat. Pura Appl. (4) **37**, 347–378 (1954).

The first part of the paper deals with asymptotic integrations of the real linear equation (1) $y''+F(x)y=0$. In the first theorem, it is supposed that $F(x)=f^2(x)+f(x)g(x)$, that f (>0) is of bounded variation on $0\leqq x<\infty$ with $f(\infty)>0$, that $\int_x^\infty g\,dx$, $g_1=\int_x^\infty g\cos 2\varphi\,dx$, $g_2=\int_x^\infty g\cos 2\varphi\,dx$ exist as improper integrals with $\varphi=\int_0^x f\,dx$, that gg_1 and gg_2 are of class $L(0, \infty)$; in which case, every solution $y=y(x)\not\equiv 0$ of (1) satisfies the asymptotic relations $y=A\cos(\varphi+B)+o(1)$, $y'=-A\sin(\varphi+B)+o(1)$, as $x\to\infty$, with $A\neq 0$. (As noted by the author, this is related to a theorem of Wintner [Amer. J. Math. **71**, 853–858 (1949); MR **11**, 437]; it is contained in a recent paper by Wintner [ibid. **76**, 183–190 (1954); MR **15**, 426].) The next theorem leads to the asymptotic formula $y=A\cos(x+(4k^2-16)^{-1}\int_0^x h^2\,dx+B)+o(1)$ if $F=1+h(x)\cos kx$, where $k\neq 0$, ± 1, ± 2, $h(x)$ is of bounded variation on $0\leqq x<\infty$, h^2 is not and h^3 is of class $L(0, \infty)$. Other theorems concern the cases $F=1+h(x)\cos kx$, where $k=1$, 2 (but different conditions are imposed on h), and the cases $F=1+\cos\lambda(x)$ with $\lambda(x)$ a convex function of sufficiently rapid growth. The last part of the paper concerns the non-linear equation $y''+F(x)y^{2n-1}=0$, where n (>1) is an integer.

In the linear cases, the asymptotic formulae are derived by asymptotic integration of the non-linear differential

equations for r and θ, where $y=r\cos\theta$, $y'=-rf\sin\theta$. In the non-linear cases, the relations defining r, θ are replaced by $y=r\psi(\theta)$, $y'=r^n f\psi'(\theta)$, where $\psi=\psi(\theta)$ is the solution of $d^2\psi/d\theta^2+n\psi^{2n-1}=0$ satisfying $\psi(0)=1$, $\psi'(0)=0$.

P. Hartman (Baltimore, Md.).

Strodt, Walter. Contributions to the asymptotic theory of ordinary differential equations in the complex domain. Mem. Amer. Math. Soc., no. **13**, 81 pp. (1954). $1.50.

In this paper the author examines for large $|x|$ the asymptotic behavior of certain solutions y of differential equations of the form $p(x, y)=0$; here $p(x, y)$ is a polynomial in $y, y', \cdots, y^{(n)}$, whose coefficients are functions of x which satisfy certain conditions. These conditions are certainly satisfied if each coefficient can be written as a fraction in which both numerator and denominator are sums of monomials; a monomial means here always a function of the form $cx^{a_0}x_1^{a_1}\cdots x_k^{a_k}$, where $x_1=\log x$ and $x_{k+1}=\log x_k$ and where the exponents are real. The required conditions are also satisfied if each coefficient has a convergent expansion or an appropriate asymptotic expansion in a series of decreasing monomials. The author does not treat all the solutions of the differential equation, but only the minimal solutions, namely those which are of minimal rate of growth at infinity.

The present paper consists of two parts. In the first part the author restricts himself to approximate minimal solutions. He calls a monomial $M=M(x)$ an approximate solution of the given differential equation if for large $|x|$ the orders of magnitude of the function $p(x, M)$ and of its derivatives possess certain prescribed upper bounds. The substitution $x=e^u$, $y=ve^{mu}$, where m denotes a suitably chosen real number, transforms the given differential equation into $p_1(u, v)=0$ such that $M=cx^{a_0}x_1^{a_1}\cdots x_k^{a_k}$ with $a_0=m$ is an approximate minimal solution of the original differential equation if and only if $cu^{a_1}u_1^{a_2}\cdots u_{k-1}^{a_k}$ is an approximate minimal solution of the new differential equation $p_1(u, v)=0$. The purpose of the first part of the paper is to show under general conditions that repeated application of this transformation leads to a differential equation $p_{k+1}(z, w)=0$ with the property that the "dominating" terms on the left-hand side form a polynomial $\lambda(w)$ in w, which is not a constant; the expression "dominating" means that the roots of the equation $\lambda(w)=0$ are the approximate minimal solutions of the final differential equation $p_{k+1}(z, w)=0$. In this way the author proves that the original differential equation $p(x, y)=0$ possesses as many approximate minimal solutions as the equation $\lambda(w)=0$ has different roots; these approximate minimal solutions have the form $cx^{a_0}x_1^{a_1}\cdots x_k^{a_k}$, where the exponents $a_0, a_1, \cdots, a_k$ are uniquely determined by the indicated algorithm and where the coefficients c denote the different roots of the equation $\lambda(w)=0$. For instance, in the case

$$P(x, y)=9yy^{(6)}+10y'y^{(5)}+2y''y^{(4)}-1$$

the three transformations

$$x=e^u,\ y=ve^{3u};\quad u=e^s,\ v=te^s;\quad s=e^z,\ t=we^{\frac{1}{2}z}$$

lead to a differential equation with z as independent and w as dependent variable, such that the dominating terms in this final differential equation form the polynomial $\lambda(w)=9w^2-1$. This polynomial has two different zeros $\pm\frac{1}{3}$, so that the original differential equation $p(x, y)=0$ has two and only two approximate minimal solutions, namely $\pm\frac{1}{3}x^3x_1x_2^{1/2}$. That $M=\pm\frac{1}{3}x^3x_1x_2^{1/2}$ is an approximate solution of the differential equation $p(x, y)=0$, means in this particular case that $p(x, M)\to 0$ as $|x|\to\infty$ and that for each positive constant integer m and for each constant integer $q\geqq 0$ we have $\theta^m p(x, M)\to 0$ as $|x|\to\infty$; here θ denotes the differential operator $xx_1\cdots x_q(d/dx)$. This result implies that each derivative of $p(x, M)$ tends for $|x|\to\infty$ rapidly to zero.

In the second part of the paper the author restricts himself to differential equations of the first order, so that $p(x, y)$ is a polynomial $p(x, y, y')$ in y and y'. Let M be an approximate minimal solution of the differential equation $p(x, y, y')=0$. By prescribing the asymptotic behavior of $p(x, M, M')$ and $q(x, M, M')$, where $q(x, y, z)=\partial p(x, y, z)/\partial z$, he determines the asymptotic behavior for large $|x|$ of the exact minimal solutions of $p(x, y, y')=0$ which are approximately equal to the function $M(x)$. The indicated algorithm provides, in the case where it gives an affirmative answer, a certain cosine function whose sign decides whether there is a unique solution of $p(x, y, y')=0$ which is approximately equal to $M(x)$ or a one-parameter family of such solutions. Moreover the author examines in how far a given minimal solution, which is analytic in a certain sector, yields by analytic continuation a minimal solution in another sector.

J. G. van der Corput (Berkeley, Calif.).

Heywood, Philip. On the asymptotic distribution of eigenvalues. Proc. London Math. Soc. (3) **4**, 456–470 (1954).

The author considers singular eigenvalue problems associated with (1) $y''+(\lambda+q(x))y=0$ on $0\leqq x<\infty$ under the rather heavy assumptions that q has a continuous second derivative, $q(0)=0$, $q'>0$, $q''\geqq 0$ for large x, $q^{-1/2}$ is of class $L(1, \infty)$, $q''=O(q'^\gamma)$ where $1<\gamma<4/3$, and $(q'/q^\alpha)'\leqq 0$ for large x where $\alpha<3/2$. In particular, (1) is of the limit-circle type. The author obtains asymptotic formulae for the number $E(\mu)$ and $F(\mu)$ of eigenvalues λ satisfying $0\leqq\lambda\leqq\mu$ and $0>\lambda\geqq-\mu$, respectively. The method depends on an adaptation of procedures of Hartman [J. London Math. Soc. **27**, 492–496 (1952); MR **14**, 278] to show that if $\lambda>0$, then $\pi N(\lambda, X)=\int_0^X(\lambda+q)^{1/2}dx+O(1)$ uniformly for large λ and X, where $N(\lambda, X)$ is the number of zeros of a solution $y=y(x, \lambda)\not\equiv 0$ of (1) on $0<x\leqq X$. An analogous formula is deduced for $N(-\lambda, X)$ if $\lambda>0$. Using the residue calculus methods of Titchmarsh, the author concludes that $\pi E(\lambda)=\int_0^\infty\{(\lambda+q)^{1/2}-q^{1/2}\}dx+O(1)$ as $\lambda\to\infty$ (and obtains a similar formula for $F(\lambda)$). A simpler procedure for deducing the formula for $E(\lambda)$ from that of $N(\lambda, X)$ follows from Hartman [Amer. J. Math. **71**, 915–920 (1949); MR **11**, 438].

P. Hartman (Baltimore, Md.).

Have

★Naĭmark, M. A. Lineĭnye differencial'nye operatory. [Linear differential operators.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 351 pp. 15.75 rubles.

This work is an important contribution to the literature on the spectral theory of ordinary linear differential operators. (The title is somewhat misleading in that partial differential operators are not analysed.) Starting from simple facts about boundary-value problems, the author develops the theory of expansion by eigenfunctions, and the spectra of ordinary differential operators, including many of the results obtained recently by Russian mathematicians. The book is divided into two parts. The first part, consisting of three chapters, contains the elementary theory of differential operators, and requires very little background on the part of the reader other than the elements of complex vari-

able theory. The last five chapters, comprising the second part, are devoted to differential operators viewed as symmetric operators in Hilbert Space, and require a knowledge of Lebesgue integration.

The first chapter develops the fundamental concepts relating to ordinary linear differential expressions of the nth order considered on finite intervals, such as boundary-value problems, adjoint problems, eigenvalues, eigenfunctions, and Green's functions. In Ch. II a complete exposition is given of the work by G. D. Birkhoff on the asymptotic behavior of eigenvalues and eigenfunctions for a not necessarily self-adjoint problem. Also included in this chapter is a treatment of the expansion theorem by the Cauchy integral method in the case of regular boundary conditions (G. D. Birkhoff), and the expansion theorem due to M. V. Keldyš for the case of separated boundary conditions. The results for a self-adjoint problem are only stated, with an appeal to the Hilbert-Schmidt theory. In Ch. III it is shown how the ideas and results developed in the first two chapters can be generalized to the case of an nth order differential expression operating on vectors.

The first chapter of the second part of the book (Ch. IV) gives basic concepts on linear operators in Hilbert space. The spectral theorem for self-adjoint operators is explained in detail, but no proof is given. Completely continuous operators are discussed, and a full treatment of the problem of extending a symmetric operator to a self-adjoint one is presented, along with certain results concerning the spectra of self-adjoint extensions of a given symmetric operator. The case of a half-bounded operator is also discussed. Ch. V is devoted to showing how a formally self-adjoint differential expression of even order (with real coefficients only) can be considered as an operator in Hilbert space, and various consequences of this. The regular and singular cases are distinguished, and quasi-derivatives are introduced. The latter notion is used to transform the given differential equation into a system of the first order which has a coefficient matrix with certain symmetry. Self-adjoint extensions of a given symmetric differential operator L_0 are characterized in terms of boundary conditions, and certain information on the deficiency index of L_0 is given. The resolvents of self-adjoint extensions of L_0 are determined, and general results on the spectra of these extensions are presented.

Ch. VI first has a discussion of spectral multiplicity for self-adjoint operators, with particular emphasis on finite multiplicity. Then the expansion theorem is proved using the method of directed functionals due to M. G. Krein. The simple proof given by B. M. Levitan is also included. The inverse transform theorem, the representation of the resolvent in terms of the spectral matrix, and the representation of the spectral matrix in terms of Green's function, are proved. Several examples in the second-order case are worked out in detail. In Ch. VII detailed results on the nature of the deficiency index and spectra of differential operators are given. Many of these are due to Russian mathematicians, including the author. They depend heavily on the asymptotic form of the solutions of linear systems, and the necessary information concerning this is developed. The final chapter presents the solution of the inverse Sturm-Liouville problem for second-order operators which is due to I. M. Gelfand and B. M. Levitan. There is a supplement on the inversion formula of Stieltjes, and a large bibliography.

E. A. Coddington (Los Angeles, Calif.).

Partial Differential Equations

Have

***Tricomi, Francesco G. Lezioni sulle equazioni a derivate parziali.** Corso di analisi superiore, anno accademico 1953–1954. Editrice Gheroni, Torino, 1954. 484 pp.

This book provides an excellent introduction to the theory of partial differential equations.

The first hundred pages contain an account of the classical analytical methods, including the theory of integral equations and the special functions of analysis, which are needed in the sequel. Ch. II contains, in 75 pages, a discussion of the theory of characteristics for equations of the first and second orders. In Ch. III (about 100 pages), the equation of hyperbolic type is discussed in detail. Equations of elliptic type are dealt with in Ch. IV (again about 100 pages), which includes not only the classical treatment, but also more modern numerical methods. The last chapter (about 100 pages) is devoted to equations of parabolic type and of mixed type.

The book is particularly valuable because of the full references to the literature of the subject and also because of the interesting sets of problems at the ends of each chapter. It is very regrettable that the book has had to be reproduced from typescript and that this has been done in an unsatisfactory manner. It deserves better treatment.

E. T. Copson (St. Andrews).

Have

***Heilbronn, Georges. Intégration des équations aux dérivées partielles du second ordre par la méthode de Drach.** Mémor. Sci. Math., no. 129. Gauthier-Villars, Paris, 1955. 99 pp. 1300 francs.

The equation treated is, in Monge's notation,

$$f(x, y, z, p, q, r, s, t)=0,$$

where equation and solution are holomorphic functions. The solution is a surface whose representation $x(\alpha, \beta)$, $y(\alpha, \beta)$, $z(\alpha, \beta)$ in curvilinear coördinates (α, β) is to be found. Drach's method is to reduce the solution to the discussion of a characteristic system, which in the general case is a passive system of twelve equations solved for the first derivatives of y, z, p, q, r with respect to α, β and for the second derivatives $x_{\alpha\beta}$, $t_{\alpha\beta}$. Special cases treated are the general Monge-Ampère equation (linear in r, s, t and $rt-s^2$) and its special form $rt-s^2=f(x, y, p, q)$, and the equations $f(r, s, t)=0$, $s=f(x, y, z, p, q)$, $s=f(x, y, z, p, q, r)$. Many of the equations of classical differential geometry are included among the types considered.

J. M. Thomas.

Hartman, Philip, and Wintner, Aurel. On the solutions of certain overdetermined systems of partial differential equations. Arch. Math. 5, 168–174 (1954).

This paper is concerned with differentiability properties of the solutions of overdetermined systems of partial differential equations of the form

$$f^j(x, y, u, v, u_x, u_y, v_x, v_y)=0 \quad (j=1, 2, 3),$$

and with some applications to geometry. The f^j are assumed to be functions of class C^n $(n>1)$ satisfying a certain differential inequality involving the Jacobians $\partial(f^i, f^j)/\partial(x, y)$. It is then proved that a solution $u(x, y)$, $v(x, y)$ of class C^2 also is of class C^{n+1}. If the equations $f^1=0$, $f^2=0$ are linear and constitute a certain kind of hyperbolic system or a certain kind of elliptic system, the same conclusion will be valid, when the $u(x, y)$ and $v(x, y)$ are assumed to be merely of class C^1. One of the applications of these theorems is to the smoothness of a surface $z=z(x, y)$ which satisfies a homogeneous, linear, elliptic partial differential equation of

the form $Az_{xx}+2Bz_{xy}+Cz_{yy}=0$ with coefficients of class C^n. If z is of class C^2, and if the Gaussian curvature K of the surface is a negative function of class C^n then z is of class C^{n+2}. Several other implications of this theory also are discussed. *A. Douglis* (New York, N. Y.).

Pliś, A. Characteristics of non-linear partial differential equations. Bull. Acad. Polon. Sci. Cl. III. 2 (1954), 419–422 (1955).

This paper contains an affirmative answer to a long-standing problem in the classical theory of single, first-order, nonlinear partial differential equations of the form $z_x=f(x, y, z, z_y)$. The problem is to prove that every solution $z(x, y)$ is generated by characteristic strips. In the classical formulation, both z and f have to be twice continuously differential; an attempted extension by Haar [Acta Litt. Sci. Szeged 4, 103–114 (1928)] contains an error. The present proof is ingenious and elementary, operating with an appropriately formed difference expression and the Arzela selection theorem. A consequence of the result is the existence of second partial derivatives of such a solution in the characteristic direction. The author extends the result to the case of more than two independent variables under the additional assumption that the characteristic equations have a unique solution. *P. D. Lax.*

Piaggio, H. T. H., and Holman, D. F. The expression of $Pdx+Qdy+Rdz$ in the form $du+vdw$. Math. Gaz. 39, 41–43 (1955).

Siegel, Carl Ludwig. Über die Existenz einer Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung. Math. Ann. 128, 144–170 (1954).

Let the Hamiltonian system $\dot{x}_k=H_{y_k}$, $\dot{y}_k=-H_{x_k}$ $(k=1, 2)$ be given, where H is a convergent real power series in x_1, x_2, y_1, y_2, vanishing at $(0, 0, 0, 0)$. Let the linear terms in H have the eigenvalues $\lambda_1, \lambda_2, -\lambda_1, -\lambda_2$, whereby no nontrivial relation $g_1\lambda_1+g_2\lambda_2=0$ holds, with integral coefficients g_i. It is then possible to introduce new variables $x_1^*, x_2^*, y_1^*, y_2^*$ related to x_1, x_2, y_1, y_2 by formal complex power series (without constant terms), such that in the new variables H is a power series depending only on the products $x_1^*y_1^*$, $x_2^*y_2^*$ [see G. D. Birkhoff, Dynamical systems, Amer. Math. Soc. Colloq. Publ., vol. 9, New York, 1927, Ch. 3]. The differential equations are then in a canonical form which is readily integrable. In the present paper it is established that, if the λ's are pure imaginary and λ_1/λ_2 is irrational, the formal power series "in general" diverge; more precisely, for convergence of the series the coefficients in the series for H must satisfy a countable infinity of analytic conditions; the sets of coefficients satisfying such conditions form a set of first category in the space of all sets of coefficients. The proof consists in showing that convergence would imply existence of a three-parameter family of periodic solutions, having arbitrarily large period and lying in a neighborhood of the origin, and that existence of such a family implies the analytic conditions mentioned above. *W. Kaplan.*

Tihonov, A. N., and Samarskiĭ, A. A. On discontinuous solutions of a quasilinear equation of first order. Dokl. Akad. Nauk SSSR (N.S.) 99, 27–30 (1954). (Russian)

Die Verfasser studieren das Anfangswertproblem der quasilinearen Differentialgleichung

$$\frac{\partial}{\partial x}A(t, u, x)+\frac{\partial}{\partial t}B(t, u, x)=F(t, u, x)$$

im Falle unstetiger Anfangswerte auf einer gegebenen Anfangskurve wie auch in Fällen, in welchen sich die Charakteristiken über dem betrachteten Grundgebiet schneiden. Unter diesen Voraussetzungen können auch unstetige Lösungen von Bedeutung werden. In Analogie zu den Erhaltungssätzen der Physik verwenden die Verfasser die Integralrelation

$$\int_{(C)} Adt-Bdx=\iint_{(S)} F\,dxdt$$

als "Vorstufe" der quasilinearen Differentialgleichung. Dabei bedeutet C die hinreichend glatte Berandung des willkürlichen Bereichs S, der im Innern des Bestimmtheitsgebietes der Differentialgleichung auf der (x, t)-Ebene angenommen wird. Da die Integralrelation auch nichtdifferenzierbare Lösungen gestattet, während differenzierbare Lösungen auf die Differentialgleichung zurückführen, gehen die Verfasser von der Integralrelation aus, wenn es sich um unstetige Lösungen handelt. Unstetige Lösungen der Integralrelation werden also als verallgemeinerte Lösungen der Differentialgleichung angesehen. Zur eindeutigen Definition solcher unstetiger Lösungen der Integralrelation sind zusätzliche Anfangsbedingungen notwendig. Die Verfasser beschränken sich auf Lösungen $u(x, t)$, die lediglich auf einer endlichen Anzahl differenzierbarer Kurven ein Ausnahmeverhalten zeigen, sonst aber überall stetig und differenzierbar verlaufen; auf den Ausnahmskurven wird die Existenz links- und rechtsseitiger Grenzwerte vorausgesetzt. Für $t=0$ wird die Anfangsbedingung $u(x, 0)=\varphi(x)$ gestellt, derart dass $\varphi(x)$ im Intervall (a, b) der x-Achse festgelegt ist und die erwähnte endliche Anzahl von Unstetigkeitspunkten zustandekommt. Die Funktionen A, B, F, A_u, B_u gelten als differenzierbare Funktionen ihrer Argumente; $f=A_u/B_u$ kann als Funktion von u monoton abnehmend angenommen werden. Wenn $\varphi(x)$ auf (a, b) stetig und differenzierbar ist, kann die Lösung mit Hilfe der Charakteristiken konstruiert werden. Als nächste Verallgemeinerung wird jetzt der Fall betrachtet, dass $\varphi(x)$ Eckpunkte hat. Sodann wird in $x=x_0$ eine Unstetigkeit zugelassen, so dass $\varphi(x_0-0)<\varphi(x_0+0)$ gilt und die "Winkelkoeffizienten" k_1, k_2 der Charakteristiken durch x_0 unstetig ausfallen. Wendet man unter diesen Bedingungen die Integralrelation auf das Rechteck $t_1\leqq t\leqq t_2$, $\xi(t_1)<x<\xi(t_2)$ an, so ergibt sich für $d\xi/dt$ der Ausdruck $\pm(A_{\text{r}}-A_{\text{II}})(B_{\text{r}}-B_{\text{II}})^{-1}$ wobei A_{r}, A_{II} und B_{r}, B_{II} den linken und rechten Grenzwerten der Kurve $\xi(t)$ in ihren Unstetigkeitspunkten entsprechen. Das positive Zeichen gilt für anwachsende Funktionen $\xi(t)$, das negative für abnehmende Funktionen $\xi(t)$. Anschliessend wird das Verhalten von $d\xi/dt$ noch in einigen weiteren Fällen von Unstetigkeiten diskutiert. *M. Pinl* (Köln).

Ciorănescu, Nicolae. Une classe d'équations linéaires aux dérivées partielles du second ordre intégrables à l'aide d'un système d'équations différentielles ordinaires. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 5, 351–360 (1954). (Romanian. Russian and French summaries)

Let E be a linear partial differential operator of the second order in two independent variables r and θ, and let the coefficients be entire functions of r and independent of θ. The author considers solutions of the partial differential equation $Eu=0$ of the form $f(r)\cos n\theta+g(r)\sin n\theta$, and investigates those cases which lead to ordinary differential equations for f and g having the origin as a regular singularity. *A. Erdélyi* (Pasadena, Calif.).

Have

✶Bergman, Stefan, and Schiffer, M. M. Properties of solutions of a system of partial differential equations. Studies in mathematics and mechanics presented to Richard von Mises, pp. 79–87. Academic Press Inc., New York, 1954. $9.00.

The authors consider the system of partial differential equations (1) $\psi_{zz^*}=F(z, z^*)$, $\psi_{ww^*}=G(w, w^*)$, where $\psi=\psi(z, z^*, w, w^*)$ and z, z^*, w, w^* are complex variables. By a method of successive approximations it is shown that there exists a solution of (1) which is analytic in the eight-dimensional product domain obtained from the domains of regularity of F and G, and which coincides with a prescribed function of the special form

$$\psi_0=A(z, w)+B(z, w^*)+C(z^*, w)+D(z^*, w^*)$$

if one of the four conditions $z=w=0$, $z=w^*=0$, $z^*=w=0$, $z^*=w^*=0$ is satisfied.

In the applications, a system of the type (1) is likely to be encountered in a real-variable problem in which the variables z^* and w^* are the complex conjugates of the variables z and w, respectively. The authors show the solutions of this real problem can always be analytically continued into an 8-dimensional "complex hull" in which the theory indicated above becomes applicable. With the help of a generalized version of Bergman's operator method it is further shown that in the real case the solutions of (1) can be represented by means of integral operators depending on two analytic functions of two complex variables.

Z. Nehari (Pittsburgh, Pa.).

Have

✶Bergman, S. On multivalued solutions of linear partial differential equations. Contributions to the theory of partial differential equations, pp. 63–68. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

This is largely an expository paper dealing with the application of the author's integral operator method [see Trans. Amer. Math. Soc. **73**, 1–34 (1952), and literature quoted there; MR **14**, 382] to the partial differential equation $\Delta u+Fu=0$, where F is an analytic function of the variables.

Z. Nehari (Pittsburgh, Pa.).

Bergman, Stefan. Essential singularities of solutions of a class of linear partial differential equations in three variables. J. Rational Mech. Anal. **3**, 539–560 (1954).

Let Δ denote the three-dimensional Laplace operator and let $F(r^2)$ be an entire function of $r^2=x^2+y^2+z^2$. This paper investigates the nature of the essential singularities of functions $\psi=\psi(x, y, z)$ which are solutions of the partial differential equation (*) $\Delta\psi+F(r^2)\psi=0$. The main tool is the author's method of integral operators which is employed to generate solutions of (*) out of functions of two complex variables via harmonic functions in three real variables. The results are too complicated for a short summary.

Z. Nehari (Pittsburgh, Pa.).

Boulanger, J. Remarques sur certaines équations aux dérivées partielles du second ordre de la forme $\Delta u+A(x, y)u=0$. Mathesis **62**, 89–99 (1953).

Soit D un domaine borné de R^2 (coordonnées x, y), de frontière C régulière. Soit v solution de (1) $\Delta v+Av=0$, Δ=Laplacien, A=fonction continue dans D, v étant nulle sur C. Si A est $\leqq 0$, ceci entraine $v=0$. Si A est >0, il peut exister des solutions non nulles. L'auteur cherche des classes de fonctions A et de domaines D, tel que $A>0$, et que (1), avec $v|_C=0$, n'admette que la solution nulle. Pour cela, soit U solution quelconque de

$$(2)\qquad \frac{\partial}{\partial x}\left(K\frac{\partial}{\partial x}U\right)+\frac{\partial}{\partial y}\left(K\frac{\partial}{\partial y}U\right)+\eta U=0,$$

$K>0$ dans $\bar{D}$, deux fois continûment différentiable dans $\bar{D}$, $\eta\leqq 0$ dans $\bar{D}$. Posons $V=U\sqrt{K}$; alors U est solution de (1) avec

$$(3)\qquad A=(4K\eta+K_x^2+K_y^2-2K\Delta K)/4K^2.$$

Les conditions: $V|_C=0$ et $U|_C=0$ sont équivalentes. Or (2) et $U|_C=0$ entraine $U=0$. Donc, si A est donné par (3), alors (1) avec $v|_C=0$ n'admet que la solution nulle. On peut choisir K et η de sorte que $A>0$, d'où une famille de solutions du problème initial. Applications avec: $K(x, y)=\alpha x+\beta y+\gamma$, $K=\alpha x^2+\beta xy+\gamma y^2$, etc.

J. L. Lions (Nancy).

Maurin, K. Der Fundamentalsatz über schwache Lösungen der allgemeinen linearen Systeme der elliptischen Differentialgleichungen beliebiger Ordnung. Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 457–461 (1955).

Dans un domaine D de R^n on donne le système différentiel qui au vecteur fonction $u=(u_1, \cdots, u_r)$ fait correspondre $Au=(Au_i)$ donné par $Au_i=\sum_{\nu=1}^{r}\sum_{|p|\leqq s} a_{pi\nu}D^p u_\nu$, les $a_{pi\nu}$ étant des fonctions données dans D. On suppose le système elliptique au sens suivant: la matrice $(a_{i\nu}(x))$, où $a_{i\nu}(x)=\sum_{|p|=s} a_{pi\nu}(x)\xi^p$, n'est pas singulière dans D. L'auteur donne des propriétés très précises de régularité de la paramétrix de l'opérateur A, selon les propriétés de régularité des $a_{pi\nu}$.

J. L. Lions (Nancy).

✶Browder, F. E. Strongly elliptic systems of differential equations. Contributions to the theory of partial differential equations, pp. 15–51. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

Let

$$K=\sum_{|\alpha|\leqq 2m} a_\alpha D^\alpha \quad (D^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}; \quad |\alpha|=\alpha_1+\cdots+\alpha_n)$$

be a linear differential operator in n variables whose coefficients $a_\alpha=a_\alpha(x)$ are linear transformations of an r-dimensional euclidean space into itself. It is said to be strongly elliptic [Višik, Mat. Sb. N.S. **29**(71), 616–676 (1951); MR **14**, 174] if its characteristic form

$$\sum_{|\alpha|=2m}(a_\alpha+a_\alpha{}^t)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n},$$

where t denotes the adjoint, is a positive definite matrix for every x and all real $\xi\neq 0$. The basic result of Višik [loc. cit.], namely that Dirichlet's problem for K and a bounded region D is solvable in the Fredholm sense if the coefficients of K are sufficiently well-behaved, is proved with some additions. In the first place, by proving an inequality announced by the reviewer [C. R. Acad. Sci. Paris **233**, 1554–1556 (1951); MR **14**, 174] Višik's restriction that D be small is removed. It is also shown that if a_α is of class $C^{|\alpha|+M}$ $(M=[(n+1)/2])$, the boundary function of class C^{2m} and the source function of class C^M, then the solution is of class C^{2m}. The existence of Green's function is established and its differentiability properties are investigated.

L. Gårding (Lund).

★**John, F. Derivatives of solutions of linear elliptic partial differential equations.** Contributions to the theory of partial differential equations, pp. 53–61. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

By the method of spherical means it is shown that a continuous weak solution of an elliptic differential equation with sufficiently differentiable coefficients possesses derivatives of arbitrarily high order. The paper is a condensed version of a more complete one [Comm. Pure Appl. Math. 6, 327–335 (1953); MR 15, 431]. *L. Gårding* (Lund).

Schaefer, Helmut. Über einen allgemeinen Konvergenzsatz von A. Korn. Arch. Math. 6, 132–135 (1955).

The last section of Korn's paper on elliptic equations in the small [Schwarz Festschrift, Berlin, 1914, pp. 215–229] contains a convergence theorem which asserts: If a sequence of functions f_n defined in a domain and vanishing on its boundary has the property that the $C^{2+\lambda}$ norm of f_n, i.e. the maximum of f, its first and second derivatives and the Hölder constant of its second derivative, grow at most exponentially in n, while the Dirichlet integral tends to zero exponentially in n, then the $C^{2+\lambda}$ norm of f_n also tends to zero exponentially. The author of the present paper points out that this theorem is in error, presenting a counterexample consisting of a sequence of increasingly rapidly oscillating functions. He further points out that the theorem can be fixed up by placing some limitation on the oscillation of f_n. Specifically, he shows that it suffices to require that $\limsup_{n\to\infty} \{|f_n|_{2+\lambda}\}^{1/n}$ is equal to $\limsup_{n\to\infty} \{|f|_2\}^{1/n}$; here $|f|_{2+\lambda}$, $|f|_2$ denote the $C^{2+\lambda}$ and C^2 norms respectively. *P. D. Lax* (New York, N. Y.).

Karabegov, V.-K. I. On stability in a closed region of Dirichlet's problem for linear equations of elliptic type. Akad. Nauk Armyan. SSR. Dokl. 16, 65–71 (1953). (Russian. Armenian summary)

Let the coefficients a_{ij} and b_i of the equation

$$(1)\qquad \sum_{i,j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{3} b_i\frac{\partial u}{\partial x_i}=0$$

be, respectively, in class $C^{(''')}$ and class $C^{('')}$ in a bounded closed region S. Let D represent a region which is properly interior to S. It is assumed that every point of the boundary Γ of D is a limit of points in the complement of D. Let $\{D_n\}$ represent a sequence of domains such that (i) $D_j \supset D_{j+1}$, (ii) $\prod_{n=1}^{\infty} D_n = \bar{D}$, and (iii) the Dirichlet Problem for (1) can be solved in each D_n for arbitrary continuous boundary data. (Such a sequence can always be found.) A given function $f(P)$ continuous on Γ can be prolonged continuously throughout S. This gives rise to continuous boundary data for each D_n and hence to a sequence of solutions $U_{n,f}$ of (1) in D_n.

Following Keldych [Uspehi Mat. Nauk 8, 171–231 (1941); MR 3, 123], the Dirichlet Problem for (1) is called stable in D if, for any continuous f, the sequence $U_{n,f}$ always converges uniformly to f on Γ. A point Q of Γ is called a point of stability relative to (1) if, for any continuous f, $\lim_{n\to\infty} U_{n,f}(Q)=f(Q)$. The author proves the theorems: I) a point Q of Γ is a point of stability relative to (1) if and only if it is a point of stability relative to the Laplace equation $\sum_{i=1}^{3} \partial^2 u/\partial x_i^2=0$; and II) the Dirichlet Problem for (1) is stable in D if and only if every boundary point is a point of stability. The proofs are based on a criterion of stability introduced by Keldych [loc. cit.]. *R. Finn* (Los Angeles, Calif.).

Simoda, Seturo. Sur théorème d'existence dans les problèmes aux limites pour l'équation $\Delta u = F(x, u, \text{grad } u)$. Osaka Math. J. 6, 243–268 (1954).

This paper presents existence theorems for the equation of the title in which Laplace's operator Δ is generalized. Let d denote a bounded region in Euclidean space of n ($\geqq 2$) dimensions, with coordinates $x=(x_i)$, for which Green's function for Δ exists. Let C^n (or L^1) denote the space of functions u having in d continuous derivatives of order n (or summable in d). Let $\overline{\Delta}$ and $\underline{\Delta}$ denote functional operators generating functions defined on d from functions belonging respectively to classes $\bar{Z} \subset C$ and $\underline{Z} \subset C$ in such a way that: a) if $u \in C^2$ then $u \in \bar{Z} \cap \underline{Z}$ and $\overline{\Delta}u=\underline{\Delta}u=\Delta u$; b) if $f \in C \cap L^1$ and $u=\int fe\, d\xi$ (where $e(x, \xi)$ is the fundamental solution of $\Delta u=0$), then $u \in \bar{Z} \cap \underline{Z}$ and $\overline{\Delta}u=\underline{\Delta}u=f$; c) if either $u=C^2$ or $u=\int fe\, d\xi$ with $f \in C \cap L^1$ and if $v \in \bar{Z}$ (or $\underline{Z}$) then $v-u \in \bar{Z}$ (or $\underline{Z}$) and $\overline{\Delta}(v-u)=\overline{\Delta}v-\overline{\Delta}u$ (or $\underline{\Delta}(v-u)=\underline{\Delta}v-\underline{\Delta}u$); and d) if $u \in \bar{Z}$ (or $\underline{Z}$) has a minimum (or maximum) at $x_0 \in d$ then $\overline{\Delta}(x_0) \geqq 0$ (or $\underline{\Delta}(x_0) \leqq 0$). The classes $\bar{Z}$ and $\underline{Z}$ and the operators $\overline{\Delta}$ and $\underline{\Delta}$ are otherwise unspecified. Two examples of operators $\overline{\Delta}$ ($\underline{\Delta}$) are given, both involving lim inf (lim sup); one is based on a difference-approximation to $\partial^2 u/\partial x_i^2$ and the other is based on averages over surfaces of spheres. Now let $S \subset C$ be convex and closed (under locally uniform convergence), and let h be a given function harmonic in d and continuous on the closure $\bar{d}$ of d. Under rather natural restrictions on F it is shown, using a modified form of Tychonoff's fixed-point theorem, that there exists a $u \in S \cap \bar{Z} \cap \underline{Z}$ such that

$$(\mathrm{E})\qquad \overline{\Delta}(u+h)=\underline{\Delta}(u+h)=F(x, u+h, \text{grad }(u+h)).$$

In a corollary it is assumed, inter alia, that functions $\underline{w}$ and $\bar{w}$ exist on $\bar{d}$ with $\underline{w} \leqq \bar{w}$ on $\bar{d}$ and $\underline{w} \leqq h \leqq \bar{w}$ on the boundary b^* of d, and it is shown that there exists a function u satisfying (E), agreeing with h on d^*, and such that $\underline{w} \leqq u+h \leqq \bar{w}$ in d. Uniqueness is not discussed. *F. A. Ficken* (Knoxville, Tenn.).

Ehrling, Gunnar. On a type of eigenvalue problems for certain elliptic differential operators. Math. Scand. 2, 267–285 (1954).

Soit un ouvert borné $\Omega \subset R^n$ satisfaisant à diverses conditions de régularité, à savoir: $\dot{\Omega}$ est une surface assez régulière; il existe un secteur sphérique donné tel que tout point $x \in \bar{\Omega}$ soit le sommet d'un secteur égal contenu dans $\bar{\Omega}$; Ω est normal au sens de R. Courant.

L'auteur établit d'abord des estimations intéressantes. Posons

$$N^k(f, g)=\int_\Omega \Big(\sum_{\Sigma\nu_i=k} D^\nu f\cdot D^\nu g\Big)dx, \quad D^\nu=\partial^{\Sigma\nu_i}/\partial x_1^{\nu_1}\cdots\partial x_n^{\nu_n},$$

$$N_t^k(f, g)=N^k(f, g)+tN^0(f, g).$$

Alors, quand t est suffisamment grand, on a, pour tout $f \in C_\infty(\Omega)$ admettant avec toutes ses dérivées une extension continue à $\bar{\Omega}$,

$$N^k(f, f)=O(t^{-(1-k/m)})N_t^m(f, f) \quad (0 \leqq k \leqq m),$$

$$|D^\nu f(x)|^2=O(t^{-[1-\frac{1}{2}(n+2\Sigma\nu)/m]})N_t^m(f, f),$$

$$\int_S |D^\nu f(x)|^2 dS=O(t^{-[1-\frac{1}{2}(n-j+2\Sigma\nu)/m]})N_t^m(f, f),$$

si l'exposant de t est négatif; dans la dernière formule, S désigne une multiplicité à $j<n$ dimensions, assez régulière et contenue dans $\bar{\Omega}$.

L'auteur applique ensuite ces inégalités à l'étude d'un problème de vibration. Dans l'espace de Hilbert $\mathfrak{H}_m$ des

fonctions dont les dérivées généralisées d'ordre $\leqq m$ sont dans $L^2(\Omega)$, il introduit la forme bilinéaire

$$U(f, g)=V(f, g)+R(f, g),$$

où

$$V(f, g)=\int_\Omega \sum_{\Sigma\mu=\Sigma\nu=m} a_{\mu\nu}(x)D^\mu f D^\nu g\, dx,$$

la matrice $(a_{\mu\nu}(x))$ étant hermitienne, bornée et uniformément définie positive dans $\bar{\Omega}$, telle que

$$d^{-1}\sum_{\Sigma\mu=m}|\xi^\mu|^2 \geqq \sum_{\Sigma\mu=\Sigma\nu=m} a_{\mu\nu}(x)\xi^\mu\bar{\xi}^\nu \geqq d\sum_{\Sigma\mu=m}|\xi^\mu|^2,$$

pour tout

$$x \in \Omega \quad \text{et} \quad \xi_1, \cdots, \xi_n \in C$$

et où $R(f, g)$ est une forme hermitienne telle que

$$|R(f, g)|=o(t)[v(f, f)+t\|f\|^2{}_{L^2}]\cdot[v(g, g)+t\|g\|^2{}_{L^2}].$$

D'autre part, il considère différents sous-espaces hilbertiens $\mathfrak{H}$ de $\mathfrak{H}_m$ formés des fonctions $f \in \mathfrak{H}_m$ qui vérifient en un sens généralisé des conditions aux limites consistant à annuler sur $\dot{\Omega}$ des relations linéaires du type

$$\sum_{\Sigma\nu<m-1} c_\nu(x)D^\nu f=0.$$

L'auteur étudie les fonctions propres φ et les valeurs propres λ définies par $\varphi \in \mathfrak{H}$ et $U(\varphi, h)=\lambda(\varphi, h)_{L^2}$ pour tout $h \in \mathfrak{H}$. Dans ce but, il démontre l'existence, pour t assez grand, d'un opérateur G_t de L^2 dans $\mathfrak{H}$, tel que

$$U(G_t f, h)+t(G_t f, h)_{L^2}=(f, h)_{L^2}$$

pour tout $f \in L^2$ et $h \in \mathfrak{H}$. Les fonctions propres sont alors les $\varphi \in L^2$ telles que $G_t\varphi=\varphi/(\lambda+t)$. L'opérateur G_t est hermitien défini positif et complètement continu de $\mathfrak{H}$ dans $\mathfrak{H}$. D'où l'existence d'un système complet de fonctions propres orthonormées dans L^2, les valeurs propres correspondantes étant réelles et positives.

Si $2m<n$, cas dans lequel les fonctions de $\mathfrak{H}_m$ sont continues, l'auteur montre que l'opérateur G_t admet un noyau. Il part de ce fait pour étudier le comportement asymptotique des valeurs et fonctions propres selon la méthode de T. Carleman en adoptant la technique de L. Gårding [Math. Scand. 1, 237–255 (1953); MR 16, 366]. *H. G. Garnir.*

***Ehrling, Gunnar. Asymptotic relations for eigenvalues and eigenfunctions for a simple vibration problem.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 26–33 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution). (Swedish)

Let D be an open bounded subset of real n-space and consider the quadratic forms

$$N^k(f, f)=\int_D \sum_{|\mu|=k}|D^\mu f|^2\, dx, \quad (f, f)=\int_D |f|^2\, dx$$

and

$$N_t{}^k(f, f)=N^k(f, f)+t(f, f),$$

where $D^\mu=(\partial/\partial x_1)^{\mu_1}\cdots(\partial/\partial x_n)^{\mu_n}$, $|\mu|=\mu_1+\cdots+\mu_n$ and $t>0$. It is shown that inequalities of the type

$$N^k(f, f)=O(t^{-(m-k)/m})N_t{}^m(f, f)$$

($k<m$, $t\to\infty$, O independent of f), are true for all sufficiently differentiable functions in D, provided the boundary of D is a suitable Lipschitz manifold. Similar estimates in terms of $N_t{}^m(f, f)$ and a negative power of t are also derived for $|D^\nu f(x)|^2$ and $\int_{S_j}^{\prime}|D^\nu f(x)|^2\, dS_j$, where S_j is a Lipschitz manifold of dimension j in D, provided $|\nu|$ and $n-j$ are sufficiently small. These inequalities without the dependence on t are due to Friedrichs [Math. Ann. 98, 205–247 (1927)].

Applications of these inequalities are made to a certain vibration problem, i.e. finding the eigenvalues and eigenfunctions of a certain quadratic form $U=V+R$ with respect to the unit form $E(f, f)=(f, f)$ and a class of functions H. The form V is supposed to be given by

$$\int_D \sum_{|\mu|=|\nu|=m} a_{\mu\nu}(x)D^\mu f\overline{D^\nu f}\, dx,$$

where the coefficients are bounded and the integrand is uniformly positive definite, $R(f, f)$ is supposed to be uniformly small with respect to $N_t{}^m(f, f)$ as $t\to\infty$ (examples of such forms founded on the previous inequalities are given), and H is a class of functions which is closed with respect to the metric $N_1{}^m(f, f)^{1/2}$ and includes all functions vanishing outside compact subsets of D (or compact subsets of an open subset of D whose measure equals the measure of D). It is shown that, if t is large enough, the equality

$$(f, g)=U_t(G_t f, g) \quad (U_t=U+tE,\ f \in L^2(D),\ g, G_t f \in H)$$

defines a completely continuous operator G_t from $L^2(D)$ to $H\subset L^2(D)$. Its eigenfunctions and eigenvalues are $\{\varphi\}$ and $\{(\lambda+t)^{-1}\}$ respectively if $\{\varphi\}$ and $\{\lambda\}$ are those of the vibration problem.

Finally, when $2m>n$ and the coefficients $a_{\mu\nu}$ of V satisfy a Lipschitz condition and R is symmetric (so that G_t is self-adjoint), the asymptotic formulas

$$\sum_{\lambda_i<t} 1=t^{n/2m}(2\pi)^{-n}\int_D V_y dy(1+o(1))$$

and

$$\sum_{\lambda_i<t}\varphi_i(x)\overline{\varphi_i(y)}=t^{n/2m}(2\pi)^{-n}V_y(\delta_{xy}+o(1))$$

are deduced by Carleman's well-known method. Here the set $\{\varphi_i\}$ is supposed to be orthonormalized and complete, $\delta_{xy}=0$ when $x\neq y$ otherwise and

$$V_y=\int_{p<1} d\xi \quad (p=\sum a_{\mu\nu}(y)\xi_1{}^{\mu_1+\nu_1}\cdots\xi_n{}^{\mu_n+\nu_n}).$$

L. Gårding (Lund).

Bers, Lipman. Non-linear elliptic equations without non-linear entire solutions. J. Rational Mech. Anal. 3, 767–787 (1954).

The equation of minimal surfaces

(1) $$(1+\varphi_y{}^2)\varphi_{xx}-2\varphi_x\varphi_y\varphi_{xy}+(1+\varphi_x{}^2)\varphi_{yy}=0$$

has the following properties. (a) Every entire solution is a linear function (a solution is entire if it is defined in the whole (x, y)-plane). (b) A single-valued solution defined in the neighborhood of the point at infinity is always such that its gradient attains a finite limit at infinity. This is even true for a multiple-valued solution, provided that its gradient is finitely many valued. (c) At a finite point a single-valued solution can have at most a removable singularity. Theorem (a) was first proved by S. Bernstein, (b) and (c) by the author.

In the paper two classes of equations of the type

(2) $$A(\varphi_x, \varphi_y)\varphi_{xx}+2B(\varphi_x, \varphi_y)\varphi_{xy}+C(\varphi_x, \varphi_y)\varphi_{yy}=0$$

are investigated regarding validity of the above three theorems. First, the coefficients A, B and C in (2), considered as functions of $u=\phi_x$, $v=\phi_y$, are assumed to be defined in a simply connected region Ω of the (u, v)-plane with first derivatives uniformly Hölder continuous in every compact

region of Ω. Secondly, (2) is supposed to be elliptic, i.e. the quadratic differential form

(3) $$A(u,v)du^2+2B(u,v)dudv+C(u,v)\,dv^2$$

is definite in Ω, say positive definite. One can then transform (3) into the isothermic form

(3′) $$\lambda(u^*,v^*)(du^{*2}+dv^{*2})$$

and can even achieve that the image domain Ω^* of Ω is a circle $u^{*2}+v^{*2}<R^2$ $(0<R\leqq\infty)$. The eccentricity of the mapping is given by

$$E(u,v)=\frac{A+C}{2(AC-B^2)^{1/2}}.$$

Call $e(r)$ the maximum of E in the circle $u^{*2}+v^{*2}\leqq r^2$ $(r<R)$.

The equations of the first class of equations considered by the author and called by him of slowly growing eccentricity are characterized by the requirement: $e(r)=o(\log r)$ if $R=\infty$ and $e(r)=O((R-r)^{\epsilon-1})$ with some positive ϵ in case of a finite R. In this class $R=\infty$ if and only if Ω coincides with the whole (u,v)-plane.

If one rewrites (2) as a system of equations for x and y as function of u^* and v^*, one obtains it in the form

$$x_{u^*}=\tau y_{u^*}-\sigma y_{v^*}\qquad(\sigma>0).$$
$$x_{v^*}=\sigma y_{u^*}+\tau y_{v^*}$$

By elimination of y or x, respectively, equations

$$x_{u^*u^*}+x_{v^*v^*}+\alpha_0x_{u^*}+\beta_0x_{v^*}=0$$
$$y_{u^*u^*}+y_{v^*v^*}+\alpha_1y_{u^*}+\beta_1y_{v^*}=0$$

are obtained and $\alpha_0^2+\beta_0^2=\alpha_1^2+\beta_1^2$. The function

$$\chi(r)=\max_{u^{*2}+v^{*2}\leqq r^2}(\alpha_j^2+\beta_j^2)$$

is called by the author the characteristic of (2) and the equation is called of slowly growing characteristic if $\chi(r)$ satisfies the inequalities for $e(r)$ in the definition of slowly growing eccentricity.

The essential results of the paper are contained in the theorems. Theorem 1. If (2) has a finite R and slowly growing characteristic, every entire solution is linear. Theorem 2. Assume that (2) has a slowly growing eccentricity. Then it has a non-linear entire solution if and only if $R=\infty$. Theorem 3. Let (2) be of slowly growing eccentricity and $R<\infty$. If $\phi(x,y)$ is a solution in $x^2+y^2>a^2$ and the gradient $\phi_x-i\phi_y$ finitely valued, then $\lim_{x^2+y^2\to\infty}(\varphi_x+i\varphi_y)$ exists and lies in Ω. Theorem 4. Under the assumption of Theorem 3 a solution $\phi(x,y)$ in $0<x^2+y^2<a^2$ with finitely-valued gradient is such that $\lim(\phi_x+i\phi_y)$ exists and belongs to Ω.

In the proofs the author makes use of theorems on quasi-conformal mappings, of his theory of pseudo-analytic functions and a recent theorem of Agmon [same J. 3, 763–765 (1954); MR 16, 686]. *C. Loewner.*

Finn, Robert. On a problem of type, with application to elliptic partial differential equations. J. Rational Mech. Anal. 3, 789–799 (1954).

With an elliptic partial differential equation of the type

(1) $$A(u,v)\varphi_{xx}+2B(u,v)\varphi_{xy}+C(u,v)\varphi_{yy}=0\qquad(u=\varphi_x,\ v=\varphi_y)$$

a conformal structure of the domain Ω of the (u,v)-plane in which A, B and C are defined, given by the quadratic differential form

(2) $$A(u,v)du^2+2B(u,v)dudv+C(u,v)dv^2,$$

is intrinsically associated. That $z=\phi(x,y)$ is a solution of (1) in a domain D of the (x,y)-plane is equivalent to the statement that the conformal structure of D defined by the quadratic differential form

(3) $$A(\varphi_x,\varphi_y)dy^2-2B(\varphi_x,\varphi_y)dxdy+C(\varphi_x,\varphi_y)dy^2$$

is transformed by the gradient mapping $u=\varphi_x$, $v=\varphi_y$ into that of (2). In the case of the minimal-surface equation the conformal structure (3) is identical with that derived from the first fundamental form of the surface $z=\phi(x,y)$. It is further known that the conformal radius of (2) of the minimal-surface equation is finite. From these two facts it is clear that Bernstein's theorem that every entire solution of the minimal-surface equation is linear is equivalent to the statement that (3) is of parabolic type. From this observation one may expect similar theorems to hold if (1) does not differ too much from the minimal surface equation. Indeed, the author proves: Theorem 1: Let (1) be the Euler-Lagrange equation of a variational problem

(4) $$\delta\int\int F(u^2+v^2)\,dxdy=0,$$

where $F(u^2+v^2)$ is defined in the whole (u,v)-plane. The ellipticity of the equations is expressed by the inequality

(5) $$F'(2F''\omega^2+F')>0\quad(\omega^2=u^2+v^2,\ F'=dF/d\omega^2).$$

Let $\phi(x,y)$ be an entire solution for which

(6) $$(F-2\omega^2F')F$$

remains between two fixed positive bounds. Then the surface $z=\phi(x,y)$ has a parabolic conformal structure. (The expression (6) is constant for the minimal surface equation.) Theorem 2: Let the coefficients of (1) be defined in the whole (u,v)-plane and call the coefficients of the minimal surface equation

$$\alpha=\frac{1+q^2}{W},\quad\beta=-\frac{pq}{W},\quad\gamma=\frac{1+p^2}{W}\quad(W=(1+p^2+q^2)^{1/2}).$$

Assume that for all u, v the expression

(7) $$\frac{A\gamma+C\alpha-2B\beta}{(AC-B^2)^{1/2}}\leqq2\epsilon,$$

ϵ being a constant. Then each entire solution of (1) representing a surface of parabolic type is a linear function. By combination of assumptions (6) and (7) one is lead to a class of differential equations for which Bernstein's theorem holds. Other such classes were considered by L. Bers [see the paper reviewed above]. *C. Loewner.*

Položil, G. N. A theorem on the correspondence of boundaries and variational theorems for certain elliptic systems of differential equations. Dokl. Akad. Nauk SSSR (N.S.) 95, 927–930 (1954). (Russian)

Let $w=u+iv$ satisfy the system

(*) $$v_y=au_x+bu_y,\quad v_x=-du_x-cu_y$$

with $\delta\equiv ac-(\frac12b+\frac12d)^2$ positive and a positive. If G is a domain in the xy-plane in which a, c, δ, $|b|$ and $|d|$ are bounded, and if G^* is the map of G in the w-plane, the following theorem on the correspondence of the boundaries L and L^* of G and G^* is established: If a, b, c, d have first derivatives which satisfy a Hölder condition and if all points of L and L^* are attained in the mapping of G onto G^* by (*), then the function $w=f(z)=u+iv$ and the inverse function $z=x+iy=f^{-1}(w)$, are uniformly continuous in the closed domains $G+L$, G^*+L^*; thus the mapping gives a one-one continuous correspondence between the points of

the closed domains. This extends previous works of the author for systems (*) [same Dokl. (N.S.) **58**, 1275–1278 (1947); **63**, 615–618 (1948); MR **9**, 507; **10**, 526]. In addition certain variational theorems are stated. *M. H. Protter.*

Moisil, Gr. C. Sur les invariants des systèmes de Vecua. II. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 595–601 (1954). (Romanian. Russian and French summaries)

The note contains formal remarks about transformations of an elliptic system of two first-order linear partial differential equations for two unknown functions of two variables.
L. Bers (New York, N. Y.).

***Lax, P. D., and Milgram, A. N. Parabolic equations.** Contributions to the theory of partial differential equations, pp. 167–190. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

The object of this paper is the initial boundary-value problem for the parabolic equation $\partial u/\partial t+Lu=0$, where L is a real elliptic differential operator of order $2p$ in n variables, which is bounded from below. It is required that u vanish of order p at the boundary of a bounded region in real n-space. This problem is solved by defining a generalized (non-selfadjoint) Friedrichs extension of L, connected with Dirichlet's problem, and applying a theorem of Hille and Yosida [Yosida, J. Math. Soc. Japan **1**, 15–21 (1948); MR **10**, 462]. The differentiability properties of the solution thus obtained are investigated by means of a method given by Friedrichs [Comm. Pure Appl. Math. **6**, 299–326 (1953); MR **15**, 430]. The object and the methods of this paper are about the same as those of the paper reviewed below.
L. Gårding (Lund).

Lyance, V. È. On a boundary problem for parabolic systems of differential equations with a strongly elliptic right-hand side. Mat. Sb. N.S. **35**(77), 357–368 (1954). (Russian)

Let A be a densely defined linear closed operator on a Hilbert space H and let both A and its adjoint A^* have negative real parts in the sense that $\Re(A\varphi, \varphi)\leq 0$ when $\varphi \in D(A)$, and $\Re(A^*\psi, \psi)\leq 0$ when $\psi \in D(A^*)$. It is shown that a weak solution of the equation $\partial u/\partial t=Au+f(t)$, where $t\geq 0$, $f(t)\in H$ is continuous in t and $u(0)=\varphi\in H$ is given, is unique and given by

$$u=T(t)\varphi+\int_0^t T(t-\tau)f(\tau)d\tau,$$

where $T(t)=\lim_{k\to\infty}(kt^{-1}(kt^{-1}-A)^{-1})^k$ is a bounded and continuous operator [cf. Hille, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; MR **9**, 594]. This result is applied to the case when A is an extension of a strongly elliptic differential operator [Višik, Mat. Sb. N.S. **29**(71), 615–676 (1951); MR **14**, 174]. The object and the method of this paper are about the same as those of the paper reviewed above. *L. Gårding.*

***Rosenbloom, P. C. Linear equations of parabolic type with constant coefficients.** Contributions to the theory of partial differential equations, pp. 191–200. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

The author deals with the partial differential equation (1) $\partial u/\partial t=L(\partial/\partial x)u$, where

$$x=(x_1, \cdots, x_n),\quad \partial/\partial x=(\partial/\partial x_1, \cdots, \partial/\partial x_n),$$
$$L=L_0+L_1(x)+\cdots+L_{2k}(x),$$

$L_j(x)$ is a homogeneous polynomial of degree j with (complex) constant coefficients and $\mathrm{Re}\{(-1)^{k-1}L_{2k}(x)\}>0$ for real x. A fundamental solution of (1) is

$$K(x, t)=(2\pi)^{-n}\int \exp\{iz\cdot x+tL(iz)\}d_zV$$

and a formal solution of (1) for $t>0$ satisfying $u(x, 0)=\varphi(x)$ is $u(x, t)=T\varphi=\int\varphi(y)K(x-y, t)\,d_yV$. The paper is a resumé of some results (without proofs) on properties of the function $u=T\varphi$; in particular, it is concerned with the question in what sense and to what extent $u=T\varphi$ is a solution of (1). The results in general depend on estimates for the kernel K and its derivatives due to Ladyzhenskaya [Mat. Sb. N.S. **27**(69), 175–184 (1950); MR **12**, 709]. Most of the theorems are too long to state here; examples of parts of an existence and uniqueness theorem are the following: If

$$\int|\varphi(x)|\exp(-M|x|^{2k/(2k-1)})\,d_xV<\infty,$$

then $u=T\varphi$ exists and is a solution of (1) for $0<t<c$ (and an estimate is furnished for c in terms of L and M). If $u(x, t)$ is continuous for all x and $0\leq t<c$, $u(x, 0)=\varphi(x)$, $u(x, t)$ is a solution of (1) and

$$\int|u(x, t)|\exp(-M|x|^{2k/(2k-1)})\,d_xV\leq\text{const.}$$

for $0<t<c$ (for a specified c depending on L and M), then $u=T\varphi$. *P. Hartman* (Baltimore, Md.).

Karimov, D. H. On periodic solutions of nonlinear differential equations of parabolic type. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **5**, 30–53 (1949). (Russian)

The author establishes some existence and uniqueness theorems for nonlinear parabolic partial differential equations of the form

$$\frac{\partial z}{\partial t}-a^2\frac{\partial^2 z}{\partial x^2}=\phi(x, t)+\mu f(z),$$

where ϕ is periodic in t. The existence of a periodic solution is established by using the method of successive approximations. *R. Bellman* (Santa Monica, Calif.).

Nicolescu, Miron. L'équation itérée de la chaleur. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 243–332 (1954). (Romanian. Russian and French summaries)

The object of the paper is to study the solutions of the equation

$$(*)\qquad L_p(u)\equiv\left(\Delta-\frac{\partial}{\partial t}\right)^p u=0$$

as well as certain associated boundary-value problems. Here p is any positive integer and Δ is the Laplace operator. The proofs are carried out for $\Delta=\partial^2/\partial x^2$. Most of the theorems are direct extensions of theorems known to hold for the heat equation $L_1(u)=0$. We list only three of the many results obtained by the author. (1) Solutions of (*) can be written in the form

$$u(x, t)=\sum_{j=0}^{p-1}(t-t_0)^j u_j(x, t),$$

where each $u_j(x, t)$ is a solution of $L_1(u)=0$. (2) If for the continuous functions $f_j(x)$ there exist positive constants M and K such that $|f_j(x)|<M\exp(Kx^2)$, then in the domain $-\infty<x<\infty$, $0<t<1/4K$ there exists a solution u of (*) such that

$$\lim_{t\to 0}u(x, t)=f_0(x),\quad \lim_{t\to 0}L_s(u)=f_s(x),\quad s=1, \cdots, p-1.$$

(3) Let D be the domain bounded by the curves

$$\Gamma_1: t=0; \quad \Gamma_2: t=T, \quad \Gamma_3: x=\gamma_1(t), \quad \Gamma_4: x=\gamma_2(t),$$

with $\gamma_1(t)<\gamma_2(t)$ for $0\leq t\leq T$. The author treats the problem of finding a solution $u=u(x,t)$ in D of the equation $L_p(u)=f(x,t)$ knowing on the boundary curves $\Gamma_1, \Gamma_3, \Gamma_4$ the values of

$$u, \quad L_1(u), \cdots, L_{p-1}(u).$$

F. G. Dressel (Durham, N. C.).

Călugăreanu, G., and Rado, Fr. Sur un problème de propagation de la chaleur. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 17–30 (1954). (Romanian. Russian and French summaries)

Let D be a plane domain bounded by the closed analytic curve Γ. Assume the functions $p=p(\xi,\eta)$, $q=q(\xi,\eta)$, $r=r(\xi,\eta)$ are bounded integrable functions defined in D; and $A(\sigma)$, $B(\sigma)$ are bounded integrable functions defined on Γ with A such that $\int_\Gamma A(\sigma)d\sigma\neq 0$. The authors show that for ϵ sufficiently small there exists a function $u=u(\xi,\eta)$ which in D satisfies the equation

$$[1+\epsilon p]\left[\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}\right]+\epsilon\left[q\frac{\partial u}{\partial\xi}+r\frac{\partial u}{\partial\eta}\right]=0$$

and which on Γ meets the condition

$$\frac{du}{dn}=\epsilon[A(\sigma)u+B(\sigma)].$$

The solution u admits a representation of the form

$$u=u_0(\xi,\eta)+\epsilon u_1(\xi,\eta)+\epsilon^2 u_2(\xi,\eta)+\cdots.$$

F. G. Dressel (Durham, N. C.).

Vodička, V. Conduction of fluctuating heat flow in a wall consisting of many layers. Appl. Sci. Res. A. 5, 108–114 (1955).

Let $u(x,t)$ denote the periodic temperatures in a laminated slab bounded by the planes $x=x_0$ and $x=x_n$. The layers $x_{k-1}<x<x_k$ $(k=1,2,\cdots,n)$ have different thermal coefficients. They are in perfect thermal contact at their interfaces. The author assumes that the temperature of the face $x=x_0$ "varies sinusoidally with time" and writes $u(x_0,t)=U(x_0)\exp(i\omega t)$. He then assumes that $u(x,t)$ has the form $U(x)\exp(i\omega t)$. Evidently $U(x)$ is a complex function of the real variable x, in order that the phase of the temperature variation with time t may vary with x; but this point is not clarified in the paper. Let $w(x,t)$ denote the flux of heat in the direction of the x-axis, $w(x,t)=W(x)\exp(i\omega t)$. By combining solutions of the heat equation in successive layers with the aid of matrices, the author obtains a formula for $u(x,t)$ and $w(x,t)$ in the kth layer in terms of $U(x_0)$ and $W(x_0)$, a formula that involves the product of k square matrices each having two rows. The paper contains no comments on how both the flux and temperature variation at the same face might be measured, or adjusted relative to each other. Thermal conditions on the face $x=x_n$ are not prescribed here. *R. V. Churchill* (Ann Arbor, Mich.).

Vodička, Václav. Heat waves in multilayer cylindrical bodies. Appl. Sci. Res. A. 5, 115–120 (1955).

Let $u(r,t)$ represent the periodic temperatures in a composite cylinder of infinite length bounded by the cylindrical surfaces $r=r_0>0$ and $r=r_n$, with interfaces $r=r_k$ $(k=1,2,\cdots,n-1)$. The formula for $u(r,t)$ in any layer is derived in terms of the temperature $U(r_0)\exp(i\omega t)$ and flux $W(r_0)\exp(i\omega t)$ at the surface $r=r_0$. The problem and the results are analogous to those described in the review above for the flat laminated wall. *R. V. Churchill.*

Mickevič, N. V. Some questions of the theory of heat conduction of anisotropic rigid bodies. Ž. Eksper. Teoret. Fiz. 26, 557–561 (1954). (Russian)

The heat equation is written in the form

$$1) \qquad \frac{\partial u}{\partial t}=a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+F(x_1,x_2,x_3,t),$$

where repeated Latin subscripts are summed over the range 1 to 3. Here a_{ij} is the heat-conduction tensor and F is a given function characterizing heat source in the body under consideration. Let Σ denote the surface bounding the region in which (1) holds; then three different types of boundary conditions are considered, namely,

$$2) \quad u|_\Sigma=\phi, \qquad 3) \quad N_i a_{ij}\frac{\partial u}{\partial x_j}\bigg|_\Sigma=\psi,$$

$$4) \qquad N_i a_{ij}\frac{\partial u}{\partial x_j}\bigg|_\Sigma=-h(u|_\Sigma-\theta),$$

where ϕ, ψ, and θ are given functions and N_i is the outward pointing vector normal to Σ. The solutions for these three cases are given in terms of the Green's function associated with a certain solution of the heat problem for an anisotropic medium in a parallelpiped, free from heat sources.

C. G. Maple (Ames, Iowa).

Keilson, Julian. On diffusion in an external field and the adjoint source problem. Quart. Appl. Math. 12, 435–438 (1955).

If particles with lifetime τ diffuse in the presence of an external field $\mathbf{F}(r)$, the density $\rho(r,t)$ satisfies the equation

$$\frac{\partial\rho}{\partial t}=D\nabla^2\rho-\frac{\rho}{\tau}-\nabla\cdot(\mathbf{F}(r)\rho).$$

The probability $\gamma(r_0)$ that a particle at r_0 will reach the collector surface before decaying or being absorbed by other surfaces may be obtained as the solution to the steady-state diffusion equation with a source at r_0 with appropriate boundary conditions at the surfaces. The author points out that it may be simpler to solve the adjoint equation

$$D\nabla^2\gamma-\frac{\gamma}{\tau}+\mathbf{F}(r)\cdot\nabla\gamma=0,$$

which has no singularity to disturb any geometric symmetry available. The boundary conditions at the collector surface is nonhomogeneous and those at the secondary surfaces homogeneous. The boundary conditions are shown to be independent of the external field. *C. G. Maple.*

★Diaz, J. B. On Cauchy's problem and fundamental solutions. Contributions to the theory of partial differential equations, pp. 235–247. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

The purpose of this article is to review some of the latest results obtained by the members of the Arden House Conference on Partial Differential Equations. The author discusses first some results of M. H. Martin [Bull. Amer. Math. Soc. 57, 238–249 (1951); MR 13, 244]. Riemann's method for solving Cauchy's problem for the equation

$$L(u)=u_{xy}-au_x-bu_y=0, \quad a=a(x,y), \quad b=b(x,y),$$

is based essentially on the bilinear Lagrange differential identity

$$vL(u)-uM(v)=A_x+B_y,\quad M(v)=v_{xy}+(av)_x+(bv)_y.$$

Martin's starting point is, instead of Lagrange's identity, the following bilinear divergence identity:

$$\left(\frac{v_x}{\Phi_x}-\frac{v_y}{\Phi_y}\right)L(u)+\left(\frac{u_x}{\Phi_x}-\frac{u_y}{\Phi_y}\right)M(v)=\left(\frac{u_xv_x}{\Phi_x}\right)_y-\left(\frac{u_yv_y}{\Phi_y}\right)_x,$$

where Φ is a solution of $L(u)=0$ for which $\Phi_x\neq 0$, $\Phi_y\neq 0$ and the associate operator M is given by

$$M(v)=v_{xy}-b\Phi_x^{-1}\Phi_yv_x-a\Phi_x\Phi_y^{-1}v_y.$$

The method has been generalized by J. B. Diaz and M. H. Martin to more than two independent variables. The author discusses the case $(n=3)$:

$$u_{xx}+u_{yy}+u_{zz}-u_{tt}=0,\quad u(x,y,z,0)=f(x,y,z),$$
$$u_t(x,y,z,0)=g(x,y,z).$$

The coefficient k/t of the Euler-Poisson-Darboux equation

$$(*)\qquad \Delta u=u_{tt}+\frac{k}{t}u_t,\quad u(x_1,\cdots,x_n,0)=f(x_1,\cdots,x_n),$$
$$u_t(x_1,\cdots,x_n)=0,$$

is infinite on the plane $t=0$. For any real number k the corresponding singular Cauchy problem was first solved by A. Weinstein [C. R. Acad. Sci. Paris **234**, 2584–2585 (1952); MR **14**, 176]. A Weinstein employed his "method of recurrence" and a generalized method of descent.

J. B. Diaz and H. F. Weinberger obtained the solution of the Cauchy problem (*) using again as a starting point the same explicit formula for the solution for $k>n-1$ which was employed by Weinstein. The definite integral in this formula diverges for $k<n-1$ and their method consists in finding the analytic continuation of this integral for $k<n-1$. The odd negative integer values of the parameter k play an exceptional role in the solution of the Cauchy problem (*). A solution of the Cauchy problem for these exceptional values of k was given by Diaz and Weinberger, after Weinstein had pointed out the particular nature of polyharmonic initial values in this problem. The result is that, if $f(x_1,\cdots,x_n)$ has continuous derivatives of order not less than the maximum of the two numbers $\frac{1}{2}(n+3-k)$ and $3-k$, there is a solution of (*).

The generalized axially symmetric potential theory deals with the elliptic equation

$$(**)\qquad \frac{\partial^2u}{\partial x_1^2}+\cdots+\frac{\partial^2u}{\partial x_n^2}+\frac{\partial^2u}{\partial y^2}+\frac{k}{y}\frac{\partial u}{\partial y}=0.$$

J. B. Diaz and A. Weinstein have shown that a fundamental solution for the equation (**) with a singularity at the point $x_1=x_2=\cdots=x_n=0$, $y=b>0$ is given by the formula

$$u_b^{(k)}=\int_0^\pi\frac{\sin^{k-1}\alpha\,d\alpha}{(\sum_1 x_i^2+b^2+y^2-2by\cos\alpha)^{(k+n-1)/2}}$$

for $k>0$. A similar formula for a fundamental solution of (**), for $k\leqq 1$, is obtained by an application of the recurrence relation which holds for solutions of (**). These formulas constitute an extension to (**) of formulas for a fundamental solution of the equation $\Phi_{xx}+\Phi_{yy}+py^{-1}\Phi_y=0$ with singularity at $x=0$, $y=b>0$, which were given by Weinstein. The case $p=\frac{1}{3}$ corresponds to Tricomi's equation. Finally, some results obtained by A. Huber are discussed.

M. Pinl (Cologne).

*Leray, J. **On linear hyperbolic differential equations with variable coefficients on a vector space.** Contributions to the theory of partial differential equations, pp. 201–210. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

Dans cet article, écrit en octobre 1952, l'auteur indique sans démonstrations les résultats qu'il a obtenus dans l'étude de l'existence et des propriétés de la solution du problème de Cauchy relatif aux équations aux dérivées partielles linéaires, hyperboliques, à coefficients variables indéfiniment dérivables ou bien lipschitziens. Depuis lors, il a exposé et démontré ces résultats dans la seconde partie (Linear hyperbolic equations with variable coefficients) de son cours à Princeton [Hyperbolic differential equations, Inst. Advanced Study, Princeton, N. J., 1953; MR **16**, 139]. Le présent exposé est particulièrement clair et permet au lecteur de s'orienter facilement dans la partie citée de ce cours.

H. G. Garnir (Liège).

Fer, Francis. Forme générale des solutions singulières des équations d'onde. C. R. Acad. Sci. Paris **240**, 600–602 (1955).

The author shows that the equation

$$(\mathrm{E})\qquad Lu=\Box u+b^\rho\frac{\partial u}{\partial x^\rho}-ku=0\quad(\rho=1,\cdots,4\colon x^4=t),$$

where the b^ρ and k are regular (real or complex) functions of the x^ρ, admits an infinity of solutions of the form

$$(\mathrm{J})\qquad u(x^\rho)=\int_{-\infty}^{\tau}\omega(\theta)\frac{\mu(x^\rho,a^\rho)}{r}\,d\theta\quad(a^4=\theta),$$

where the a^ρ are the coordinates of an arbitrary trajectory of velocity v^i bounded by $\bar\omega<c$, $r^2=\sum(x^i-a^i)^2$, and τ satisfies the relation $c(t-\tau)-r(\tau)=0$; conversely, all solutions with a suitable mobile polar singularity can be represented in the form of a line potential (J). The proof is based on the method of Hadamard [Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques, Hermann, Paris, 1932]. If the representation (J) is to yield a solution of (E), then the generalized Riemann's formula, applied to the functions $\mu(x^\rho)$ and a fundamental solution of (E) in a suitable region, determines $\mu(x^\rho,a^\rho)$ in terms of an arbitrary function. Conversely, if $\mu(x^\rho)$ is a solution of (E) with an appropriate singularity, the same method leads to the representation (J). The author notes some extensions of his results.

R. Finn (Los Angeles, Calif.).

*Protter, M. H. **A boundary value problem for the wave equation and mean value theorems.** Contributions to the theory of partial differential equations, pp. 249–257. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

Let D be the domain bounded by the three surfaces

$$x^2+y^2=(z-z_0)^2,\ (x-x_0)^2+(y-y_0)^2=z^2,\ f=0\ (x_0^2+y_0^2<z_0^2).$$

The author seeks a solution of the equation

$$(*)\qquad u_{xx}+u_{yy}=u_{zz}$$

in D satisfying the boundary conditions

(**) $u(x,y,0)=\Phi_1(x,y)$ for x, y on the circle S_1: $x^2+y^2\leqq z_0^2$;

$u(x,y,z)=\Phi_2(x,y)$ for x, y, z on S_2, the part of $x^2+y^2=(z-z_0)^2$ coinciding with D,

($\Phi_1=\Phi_2$ on the circle $x^2+y^2=z_0^2$). He proves the theorem:

There exists a unique solution $u(x, y, z)$ of (*) in D satisfying the boundary conditions (**). The result allows one to establish a mean-value theorem which may be considered as a generalization of the theorem establishing the solution of the Abel integral equation. The result may be generalized in the following way: let $F(x, y)$ be continuously differentiable in the unit circle K and suppose $P_0(x_0, y_0)$ is a fixed point in K. If the mean value of F is given as a continuously differentiable function $g(x, y)$ over all circles in K which are tangent to the boundary of K and do not contain the point x_0, y_0 then $F(x, y)$ is uniquely determined. If the mean value of F is given as a continuously differentiable function $g(x, y)$ over all circles in K passing through P_0, then $F(x, y)$ is uniquely determined. *M. Pinl* (Cologne).

Gurevič, M. I. On some solutions of the wave equation. Dokl. Akad. Nauk SSSR (N.S.) **97**, 385–386 (1954). (Russian)

If the wave equation

$$\phi_{xx}+\phi_{yy}-\phi_{zz}=0 \tag{1}$$

is transformed through the substitution

$$x=-r\frac{\cos\sigma}{\sinh\delta},\quad y=-r\frac{\sin\sigma}{\sinh\delta},\quad z=-r\coth\delta,$$

it becomes

$$\phi_{\sigma\sigma}+\phi_{\delta\delta}=\frac{1}{\sinh^2\delta}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right). \tag{2}$$

The author finds solutions of equation (1) inside the cone $z^2-x^2-y^2=0$ $(z>0)$, where $r=(z^2-x^2-y^2)^{1/2}$ is real. Solutions of (1) are written in the form $\phi=\sum_{n=1}^{\infty} r^n\phi_n$, where $\phi_n(\delta, \sigma)$ must satisfy the equation

$$\frac{\partial^2\phi_n}{\partial\sigma^2}+\frac{\partial^2\phi_n}{\partial\delta^2}-\frac{n(n+1)}{\sinh^2\delta}\phi_n=0.$$

An iterative procedure is given for the determination of ϕ_n in terms of arbitrary harmonic functions. In any given physical problem, these harmonic functions are determined from the boundary conditions. *C. G. Maple.*

Jessel, Maurice. Une formulation analytique du principe de Huygens. C. R. Acad. Sci. Paris **239**, 1599–1601 (1954).

"A chaque équation de propagation, on peut associer un ensemble de formules traduisant le principe de Huygens pour une partition donnée de l'espace-support. L'exposition naïve donnée ici doit mettre en évidence le caractère général et direct de la méthode." (Author's summary.) The exposition is so brief that it is not possible to assess the value of the ideas stated. *E. T. Copson* (St. Andrews).

Slivnyak, I. M. On boundary problems for Maxwell's equations. Mat. Sb. N.S. **35(77)**, 369–394 (1954). (Russian)

The author considers Maxwell's equations rot $H=\partial E/\partial t$, rot $E=-\partial H/\partial t$ in a bounded region G in three-space under the assumption that the flow of energy $\int(E\times\bar{H})_n d\Gamma$ through the boundary Γ vanishes. The boundary conditions can then be given the very general form $E\in D(A)$, $H\in D(A^*)$, where A and its adjoint A^* are operators on the Hilbert space H of all square integrable complex vector-functions in G and A is an extension of the operator "rot" defined on all smooth vector-functions with vanishing tangential component at Γ. Following Višik [Trudy Moskov. Mat. Obšč. **1**, 187–246 (1952); MR **14**, 473], the author classifies all A which are completely solvable, i.e. for which A_0^{-1} is completely continuous, where A_0 is the restriction of A from $D(A)$ to the part of $D(A)$ which is orthogonal to the null-space of A. If A_0^{-1} is self-adjoint, the original initial problem can be solved by a Fourier expansion. *L. Gårding* (Lund).

Nardini, Renato. Sul comportamento asintotico della soluzione di un problema al contorno della magneto-idrodinamica. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 225–231, 341–348 (1954).

These notes continue earlier work by the same author [Ann. Mat. Pura Appl. (4) **35**, 269–290 (1953); MR **16**, 202]. The physical problem considered reduces to solving

$$\frac{\partial^2 H}{\partial t^2}=a^2\frac{\partial^3 H}{\partial t\partial z^2}+V^2\frac{\partial^2 H}{\partial z^2}\quad (a,\ V \text{ constants})$$

under the conditions

$$\begin{array}{lll} H\to 0,\ \partial H/\partial t\to 0 & \text{as } t\to+0, & 0<z<+0,\\ H\to G(t), & \text{as } z\to+0, & t>0,\\ H\to 0, & \text{as } z\to+\infty, & t>0,\end{array}$$

where $G(t)\equiv 0$ for $t<0$. Denote the solution by $H(t, z, a)$. Then, if $a=0$, the solution is $G(t-zV^{-1})$.

The following results are proved concerning the behaviour of $H(t, z, a)$ as $a\to 0$. I. A sufficient condition that

$$H(t, z, a)\to G(t-zV^{-1}) \text{ as } a\to 0$$

is that the Laplace transform $g(s)$ of $G(t)$ exists and satisfies $|g(s)|\leqq M|s|^{-k}$ in some half-plane Re $(s)\geqq\alpha$, where M and k (>1) are constants. II. If the third derivative $G'''(t)$ exists for $t>0$, if $G(t)$ tends to zero and $G'(t)$, $G''(t)$ to finite limits as $t\to+0$, if the Laplace transform of $G'''(t)$ is $O(s^{-h})$ in some half-plane Re $(s)\geqq\delta$ with $h>0$, then

$$H(t, z, a)=G(t-z/V)+a^2R(t, z, a),$$

where R is a bounded function of a. *E. T. Copson.*

Difference Equations, Special Functional Equations

Bačelis, R. D. On difference operators with constant coefficients. Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 69–80 (1955). (Russian)

Let L be a difference operator of the form

$$Lf(y_1, \cdots, y_n)=\sum\nolimits^{(k)} \alpha_{r_1,\dots,r_k}f(y_1-r_1, \cdots, y_k-r_k) \quad (0\leqq r_j\leqq R_j;\ 1\leqq j\leqq k), \tag{1}$$

where the coefficients are constants. The author studies the equation (2) $Lf=\psi$, where ψ is assigned; it is assumed that $\psi(y)=0$, for $y_1^2+\cdots+y_k^2>0$, and $=1$ for $y_1^2+\cdots+y_k^2=0$, which is a case of special interest. Under various conditions on L, A. Stöhr [Math. Nachr. **3**, 208–242, 295–315, 330–357 (1950); MR **12**, 711] had obtained solutions of such equations with least order of growth at infinity. The author gives a new method solving (2), applicable with all operators L of form (1); the obtained solution increases at infinity not faster than a certain polynomial in $y_1, \cdots, y_n$. *W. J. Trjitzinsky* (Urbana, Ill.).

Arešev, M. S. On a finite-difference equation. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **5**, 11–19 (1949). (Russian)

Let C_m be the class of functions continuously differentiable up to order m on the segment $[a, b]$; denote by $E(u)$ the greatest integer $\leqq u$. The author proves the following. If in

the equation (1) $F(x+h)-F(x)=f(x)$ the given function $f(x) \in C_m$, then for $h>0$ a solution of (1) is

$$F(x)=\sum_{k=0}^{r} f(x-kh)+L_a[v(x, z, h, a)]_{z=1},$$

where $r=r(x, h, a)=E((x-a)/h)$;

$$L_a=\sum_{\nu=0}^{m} \frac{1}{\nu!}[f(z)e^{-(x-a)}]_{z=\infty}^{(\nu)} \frac{\partial^\nu}{\partial z^\nu},$$

$v=\exp([x-a-hr]z)/(e^{hz}-1)$; $\Phi \in C_m$. There is a similar result for $h<0$. *W. J. Trjitzinsky* (Urbana, Ill.).

Marinescu, G. Certains aspects de la théorie des équations à différences finies. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 4, 355–371 (1953). (Romanian. Russian and French summaries)

The first part of this paper develops an algebraic theory of linear difference equations, based on a difference operator whose constants are the periodic functions. These constants play the role of coefficients, and his results are intended to be analogous to the classical result in homogeneous linear differential equation theory, that every solution can be expressed as a linear combination with constant coefficients of solutions from a certain basic set. In applying these ideas he regards a function on $[0, \infty)$ as a sequence of functions on $[0, 1]$, so that a periodic function is a constant sequence.

In the second part of the paper he proves a generalization of the theorem of Poincaré on difference equations. The theorem of Poincaré is as follows: suppose that the coefficients of the difference equation

$$(1) \qquad x(t+k)+a_1(t)x(t+k-1)+\cdots+a_k(t)x(t)=0$$

satisfy the following conditions: (i) $\lim_{t\to\infty} a_m(t)=a_m$, $a_k(t)\neq 0$, $a_k\neq 0$; (ii) the roots of the equation

$$(2) \qquad \lambda^k+a_1\lambda^{k-1}+\cdots+a_k=0$$

satisfy $|\lambda_i|\neq|\lambda_j|$ for $i\neq j$. Then if $x(t)$ is a solution of (1), $\lim_{t\to\infty} x(t+1)/x(t)=\lambda_i$ for some i. The generalization of the author is as follows: Consider the same equation (1), with (i) and (ii) replaced by the following:

$$(i') \qquad \lim_{t\to\infty}|a_i(t)-\alpha_i(t)|=0,$$

where the α_i are periodic functions; (ii') the moduli of the roots $\lambda_1(t), \cdots, \lambda_k(t)$ of the equation

$$(2') \qquad \lambda^k+\alpha_1(t)\lambda^{k-1}+\cdots+\alpha_k(t)=0$$

are bounded above, away from zero, and away from each other as $t\to\infty$. Then any solution of (1) must satisfy

$$\lim_{t\to\infty}\left|\frac{x(t+1)}{x(t)}-\lambda_i(t)\right|=0$$

for some i. *J. M. Danskin* (Washington, D. C.).

Berezanskiĭ, Yu. M. On expansion in eigenfunctions of partial difference equations. Dokl. Akad. Nauk SSSR (N.S.) 93, 5–8 (1953). (Russian)

Consider the "ordinary" difference operator

$$l(u)=a_{j-1}u_{j-1}+c_ju_j+a_ju_{j+1} \quad (a_j>0, \operatorname{Im} c_j=0),$$

which is defined for sequences u_j, where $j\geq 0$ is an integer. From the theory of Jacobi matrices [cf. N. I. Ahiezer, Uspehi Mat. Nauk 9, 126–156 (1941); MR 3, 110] it is known that to each operator l there corresponds a non-decreasing "spectral function" $\tau(\lambda)$, with $-\infty<\lambda<+\infty$, and a system of polynomials $P_0(\lambda), P_1(\lambda), \cdots$, which are orthogonal with respect to $d\tau(\lambda)$ and such that if f_j, g_j are two sequences which vanish for sufficiently large j, then

$$(*) \qquad \sum_{j=0}^{\infty} f_j\bar{g}_j=\int_{-\infty}^{\infty} F(\lambda)\overline{G(\lambda)}\, d\tau(\lambda),$$

where $F(\lambda)=\sum_{j=0}^{\infty} f_jP_j(\lambda)$ is the "Fourier transform" of the sequence f. For higher-order difference operators, formulas analogous to (*) are known, with the spectral function $\tau(\lambda)$ replaced by a finite spectral matrix [Ya. M. Každan, Dokl. Akad. Nauk SSSR (N.S.) 82, 329–332 (1952); MR 13, 727; M. G. Kreĭn, ibid. 69, 125–128 (1949); MR 11, 670]. The present note is concerned with the derivation of a formula similar to (*) for the partial difference operator

$$L(u)=a_{j-1,k}u_{j-1,k}+a_{jk}u_{j+1,k}+b_{j,k-1}u_{j,k-1}+b_{jk}u_{j,k+1}+c_{jk}u_{jk}, \quad (a_{jk}>0, \operatorname{Im} b_{jk}=\operatorname{Im} c_{jk}=0),$$

where again $j\geq 0$. The proof employs a generalization of a procedure of M. G. Kreĭn [Akad. Nauk Ukrain. RSR. Zb. Prac' Inst. Mat. 1948, no. 10, 83–106; MR 14, 56].

J. B. Diaz (College Park, Md.).

Berezanskiĭ, Yu. M. On the spectral matrix of a partial difference equation. Dokl. Akad. Nauk SSSR (N.S.) 97, 573–576 (1954). (Russian)

In a previous note [see the paper reviewed above] the author considered the eigenfunction expansion problem for the partial difference equation

$$(*) \qquad L(u)_{j,k}=a_{j-1,k}u_{j-1,k}+a_{j,k}u_{j+1,k}+b_{j,k-1}u_{j,k-1}+b_{j,k}u_{j,k+1}+c_{j,k}u_{j,k}=\lambda u_{j,k} \quad (a_{j,k}>0, \operatorname{Im} b_{j,k}=\operatorname{Im} c_{j,k}=0)$$

in $j\geq 0$ with the boundary condition

$$u_{-1,k}=0 \quad (k=\cdots, -1, 0, 1, \cdots).$$

Let $P_q(\alpha;\lambda)$ be the solution of (*) satisfying $P_{-1,k}(\alpha;\lambda)=0$, $P_{0,k}(\alpha;\lambda)=\delta_{k,\alpha}$. Here $\alpha=\cdots,-1,0,1,\cdots$, and q denotes a point in a right half-plane with integer coordinates. For each finite sequence $\{f_q\}$ let $\{F_\alpha\}$ be its Fourier transforms, $F_\alpha(\lambda)=\sum_q f_qP_q(\alpha;\lambda)$ $(\alpha=\cdots,-1,0,1,\cdots)$. Then the Parseval equality

$$\sum_q f_q\bar{g}_q=\sum_{\alpha,\beta=-\infty}^{\infty}\int_{-\infty}^{\infty} F_\alpha(\lambda)\overline{G_\beta(\lambda)}\, d\tau_{\alpha,\beta}(\lambda)$$

is valid, where $T=(\tau_{\alpha,\beta})$ is a non-decreasing spectral matrix function. Let $\mathfrak{H}$ be the Hilbert space of all sequences $\{f_q\}$ such that $\sum|f_q|^2<\infty$, and denote by H the subset consisting of all finite sequences $\{f_q\}$. Define the symmetric operator A by $(Af)_q=L(f)_q$ for all $f\in H$ for which $f_{-1,k}=0$. Each self-adjoint extension of A (in $\mathfrak{H}$, or in a larger space) determines a spectral matrix, and conversely. The problem (*), with $u_{-1,k}=0$, is said to be determined if the closure of A is self-adjoint; otherwise the problem is indeterminate. The author states several results which have analogues in the theory of the classical moment problem. A characterization of the deficiency index of A is given, and from this certain sufficient conditions for the boundary-value problem to be determined are obtained. The notion of an absolutely indeterminate problem is introduced, such a problem being in a certain sense an extreme case of an indeterminate one. The set of all spectral matrices in the absolutely indeterminate case is described, and a sufficient condition is given in order that the problem be absolutely indeterminate.

E. A. Coddington (Los Angeles, Calif.).

***Bellman, R., and Danskin, J. M., Jr. The stability theory of differential difference equations.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 107–123. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

This is essentially an expository survey of various stability problems, terminated by an extensive bibliography. Besides systems of ordinary differential equations the following are some of the typical equations considered:

$$u''(t)+a^2u(t)+\int_0^1 k(t-s)u'(s)\,ds=f(t) \quad \text{(theory of elasticity)};$$

$$\frac{du(t)}{dt}=au(t)+bu(t-1)+f(t) \quad \text{(trade cycles)};$$

$$u''(t)+a_1u'(t)+a_2u'(t-d)+b_1u(t)+b_2u(t-d)=f(t)$$

(control systems, with a non-negligible time lag d).

Application of Laplace transforms to linear differential-difference equations with constant coefficients is discussed, as well as asymptotic representations of solutions. The authors then present the work of Pontrjagin on the location of the zeros of functions $P(z, e^z)$, $Q(z, \cos z, \sin z)$, where P, Q are polynomials in the arguments. They finally discuss the stability theory of non-linear differential-difference equations, which presents an analogy to the corresponding Poincaré-Liapounoff stability theory for differential systems.

W. J. Trjitzinsky (Urbana, Ill.).

Cooke, K. L. Forced periodic solutions of a stable non-linear differential-difference equation. Ann. of Math. (2) **61**, 381–387 (1955).

The author studies the equation

$$(1) \quad u^{(1)}(t+1)=au(t)+bu(t+1)+f(u(t), u(t+1), \omega t)+kg(\omega t);$$

the roots of $se^s-be^s-a=0$ are assumed to have negative real parts, a, b, k, ω, t are real, $\omega>0$; real $g(t)$ is continuous, is of period 1, its average over a period being zero; real $f(x, y, t)$ is continuous in x, y, t, is of period 1 in t, $f(0, 0, t)=0$; for every $\eta>0$ there exists $\Delta=\Delta(\eta)>0$ such that $|f(x_1, y_1, t)|\leqq\eta(|x_1-x_2|+|y_1-y_2|)$ for $|x_1-x_2|<\Delta$ and for $|y_1-y_2|<\Delta$. Under the above conditions there exists a $\delta=\delta(a, b, f, g)$, such that if $|k|(1+\omega)^{-1}<\delta$ equation (1) has a continuous solution $p(t)$ of period ω^{-1},

$$|p(t)|\leqq\rho|k|(1+\omega)^{-1}$$

(some ρ, >0, independent of k and ω); there exists an $\epsilon>0$ so that every solution $x(t)$ for which $\max\,(0\leqq t<1)|x(t)|\leqq\epsilon$ satisfies $\lim\,(x(t)-p(t))=0$ $(t\to+\infty)$. The methods used are of the type found in papers of Bellman [Ann. of Math. (2) **50**, 347–355 (1949); MR **10**, 715] and of A. B. Farnell, C. E. Langenhop, and N. Levinson [J. Math. Phys. **29**, 300–302 (1951); MR **12**, 706]. *W. J. Trjitzinsky.*

Brownell, F. H., and Ergen, W. K. A theorem on rearrangements and its application to certain delay differential equations. J. Rational Mech. Anal. **3**, 565–579 (1954).

Let f be a positive, increasing, and continuously differentiable function. Let $c, \theta>0$. The authors consider the delay-differential equation

$$(1) \quad x'(t)=-\frac{c}{\theta}\int_0^\theta(\theta-h)[f(x(t-h))-1]\,dh,$$

and prove that to every continuous function $x(t)$ defined on $-\theta\leqq t\leqq0$ there corresponds a unique solution to (1), which approaches in a certain sense the set of periodic solutions of (1). Periodic solutions to (1) must satisfy for a positive constant γ the equation

$$(2) \quad \int_0^{x(t)} f(z)\,dz-x(t)+(2c)^{-1}[x'(t)]^2=\gamma,$$

from which they may be explicitly computed. The computational techniques are discussed, conditions for the finiteness on the set of periodic solutions are found, and asymptotic formulas given.

The rearrangement theorem [see Hardy, Littlewood, and Pólya Inequalities, 2d ed., Cambridge, 1952, p. 278; MR **13**, 727] is used to prove that the period of a periodic solution of (1) is a multiple of θ, and also in proving the authors' theorems about the asymptotic behavior at infinity of arbitrary solutions. *J. M. Danskin.*

Cunningham, W. J. A nonlinear differential-difference equation of growth. Proc. Nat. Acad. Sci. U. S. A. **40**, 708–713 (1954).

The author studies the nonlinear differential-difference equation

$$(1) \quad \dot{x}(t)=[a-bx(t-\tau)]x(t),$$

which describes, for example, under certain circumstances, the growth of a fluctuating population of organisms with gestation period τ, by approximating to it by the differential equation

$$(2) \quad \ddot{x}-\alpha^2\tau\dot{x}+(\alpha^2x)/(bx)+\alpha^2x=\alpha^2\beta,$$

in which $\ddot{x}$, $\dot{x}$, and x are calculated at t and α and β are equal to $2/\tau^2$ and a/b respectively. He gives conditions under which the approximate solutions he finds for (2) ought to apply to (1) as well, and presents a graphical check on these, in a particular case, gotten on an analogue computer.

J. M. Danskin (Washington, D. C.).

Bellman, Richard. Monotone approximation in dynamic programming and the calculus of variations. Proc. Nat. Acad. Sci. U. S. A. **40**, 1073–1075 (1954).

The author writes

$$T(f)(x)=\max_{0\leqq y\leqq x}[g(y)+h(x-y)+f(ay+b(x-y))],$$

where $0<a$, $b<1$, $g(0)=h(0)=0$, $0\leqq x\leqq c$, and g and h are continuous. He considers the equation $Tf=f$, and shows how by an appropriate choice of an initial "policy", i.e. a function y_0 on $[0, c]$ satisfying $0\leqq y_0(x)\leqq x$, an initial f_0 may be generated such that the sequence $f_0, f_1, \cdots, f_n, \cdots$, defined recursively by $f_{n+1}=Tf_n$, will satisfy the monotonicity condition $f_{n+1}(x)\geqq f_n(x)$. There is a sketch of the application of the same idea to calculus of variations problems $\int f(x, y, y')\,dx$ with the left endpoint fixed, and to eigenvalue problems. *J. M. Danskin.*

Ghermănescu, M. Un système d'équations fonctionnelles. Acad. Repub. Pop. Romîne. Bul. Ști. Secț. Ști. Mat. Fiz. **5**, 575–582 (1953). (Romanian. Russian and French summaries)

Complete solution of the functional equation

$$M(x+y)=M(x)M(y)$$

for matrix functions of order 2, whose elements are measurable functions. *J. L. Doob* (Urbana, Ill.).

Functional Analysis

★Fichera, Gaetano. Lezioni sulle trasformazioni lineari. Vol. I. Introduzione all'analisi lineare. Istituto Matematico, Università, Trieste, 1954. xvii+502+iv pp.

This volume is intended, according to its preface, to be the first of a series of three. The titles of the successive chapters are given below, with comments on points of special interest.

Chapter I: Abstract sets, theory of limits, topological spaces. The author uses the older notations $X+Y$, $X\cdot Y$, and so on for set unions, intersections, etc. In the theory of limits a notion equivalent to that of filter base appears as a 'group of sets'. The axiom of choice is avoided when possible; sometimes this is done by introducing unusual definitions, e.g. for (sequential) compactness. Chapter II: Metric spaces. Here again there is an unusual definition, that of completeness.

Chapter III: Linear sets, Fréchet linear spaces. The author gives an interesting algebraic formulation of the principle of the alternative that is applicable in many cases where the usual algebraic statement is irrelevant. There are applications of the theory of Fréchet spaces (Banach's 'spaces of type (F)') to differential and integral equations. Chapter IV: Banach spaces. The principle of the alternative developed in Chapter III is applied to linear mappings with a closed range. There are applications to existence theorems for elliptic and parabolic boundary value problems. Chapter V: Hilbert spaces. Variational problems and the Ritz method are discussed.

Chapter VI: Measure theory. The author describes a direct definition of Lebesgue measure in a Euclidean space by a single limiting operation. Chapter VII: Integration. Chapter VIII: Differentiation and absolutely continuous functions. This theory is expounded in a fairly general context. There is a section on vectors of bounded variation (or absolutely continuous) with respect to a linear transformation and a vector-valued measure; this leads to a way of treating the Laplacian, the gradient and other such operators as derivations. Chapter IX: Function spaces.

Throughout the book the author gives special attention to methods and results having applications in the theory of partial differential equations. *F. Smithies.*

Köthe, Gottfried. Zur Theorie der kompakten Operatoren in lokalkonvexen Räumen. Portugal. Math. **13**, 97–104 (1954).

Let E and F be locally convex topological linear spaces. Following Leray [Acta Sci. Math. Szeged **12**, Pars B, 177–186 (1950); MR **12**, 32] the author calls a continuous linear mapping A of E into F compact if there exists a neighborhood U of 0 in E such that the closure of $A(U)$ is compact in F. The dual operator A' is a continuous linear mapping of the dual F' into the dual E' (the dual space E' consists of all continuous linear functionals on E). Since F' and E' can be topologized in various ways, there are various senses in which A' may be compact. The topology of uniform convergence on the bounded subsets of E is called the strong topology of E'. If E is a normed space the strong topology of E' is the usual norm-topology. If E and F are Banach spaces, the operator A is compact if and only if A' is compact in the sense of the strong topologies of F' and E'. This is a theorem of Schauder [Studia Math. **2**, 183–196 (1930)]. The theorem in this form is not valid for (F) spaces (as Köthe says he knows by an unpublished result of Grothendieck). However, there is given here a generalized version of Schauder's theorem, equivalent to it for Banach spaces, but valid for arbitrary E and F (not necessarily metrisable or complete). The formulation involves topologies intermediate between the weak and the strong, and subspaces with a norm generated by a bounded subset of the space. For (F) spaces it is shown that if A is compact, A' is compact in the sense of the strong topologies, and conditions are given under which A is compact if A' is compact.

In a paper by M. Altman [ibid. **13**, 194–207 (1953); MR **15**, 436] apparently not known to Köthe when he wrote this paper, it is shown that a kind of compactness of A' follows from that of A; the definitions are not quite the same as in the present paper, however.

The paper also contains extensions of two theorems of L. Schwartz [C. R. Acad. Sci. Paris **236**, 2472–2473 (1953); MR **15**, 233] about properties of a continuous linear operator B which are shared by $A=B+C$ if C is compact. Sample: If the range of B is closed and possesses a finite-dimensional complementary subspace, the same is true of A, assuming that E and F are metrisable and complete. *A. E. Taylor* (Los Angeles, Calif.).

Wolfson, Kenneth G. A class of primitive rings. Duke Math. J. **22**, 157–163 (1955).

Soient A, B deux espaces vectoriels en dualité sur un corps (commutatif ou non), K un anneau de transformations linéaires de A, continues pour la topologie faible $\sigma(A,B)$, K contenant toutes les applications continues de rang fini. L'auteur considère dans K les idéaux qui sont des "annulets", savoir des annulateurs (à droite ou à gauche) de sous-ensembles de K, et se pose la question suivante: à quelle condition tout annulet est-il engendré par un idempotent? La réponse est que, pour la topologie $\sigma(A,B)$, tout sous-espace fermé de A doit admettre un supplémentaire topologique. La forme sous laquelle l'auteur énonce ce résultat est plus compliquée, parce qu'il n'utilise pas la topologie faible, ce qui le conduit aussi à démontrer des résultats (seconde partie du lemme 3 et lemme 6) qui sont des conséquences triviales de résultats topologiques généraux. Comme application de son théorème, l'auteur prouve également que si A est un espace de Banach, $E_0(A)$ l'anneau des endomorphismes continus de A, de rang fini, alors pour que tout idéal à droite fermé dans $E_0(A)$ (pour la topologie de la norme) soit un annulet, il faut et il suffit que A soit réflexif. *J. Dieudonné* (Evanston, Ill.).

Ghika, Al. Le prolongement des fonctionnelles générales linéaires dans les modules semi-normés. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **1** (1950), 251–281 (1951). (Romanian. Russian and French summaries)

This paper contains the proofs of the theorems announced in an earlier note [Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**. 399–405 (1950); MR **13**, 565]. *G. K. Kalisch* (Minneapolis, Minn.).

Cristescu, Romulus. Sur une notion de convergence. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 297–304 (1954). (Romanian. Russian and French summaries)

The author generalizes the notion of a K-normed linear space due to Vulich [Ann. of Math. (2) **38**, 156–174 (1937)] by replacing Vulich's K-norm for finite sequences by pseudo K-norms based on an arbitrary directed set. The axioms imposed are otherwise identical with those of Vulich. Convergence and completeness are discussed, and an example (the bounded measurable functions on [0, 1]) is examined.

[For the notion of convergence employed, see also Maddaus, ibid. **42**, 229–246 (1941); MR **2**, 221.] *E. Hewitt.*

Day, Mahlon M. Strict convexity and smoothness of normed spaces. Trans. Amer. Math. Soc. **78**, 516–528 (1955).

A normed linear space E is called "strictly convex" (SC) provided its unit sphere $\{x \in E: \|x\|=1\}$ contains no line segment, and "smooth" (SM) provided through each point of the unit sphere there passes a unique supporting hyperplane; E is called SCM provided it is both SC and SM. For $x=c$, m, or cm, E is called sx provided E is isomorphic with a space which is SX. With S denoting either of the two properties SC and SM, let S* denote the other. The notions S and S* are dual in the following sense: (i) if E^* is S*, then E is S; (ii) for a reflexive E, E^* is S* if and only if E is S, and E^* is s* if and only if E is s. This duality is similar to that between the notions of uniform convexity and uniform flattening, studied earlier by the author [Ann. of Math. (2) **45**, 375–385 (1944); MR **6**, 69].

Clarkson proved [Trans. Amer. Math. Soc. **40**, 396–414 (1936)] that every separable normed linear space is sc, and further results concerning sc and sm spaces were obtained by the reviewer [ibid. **74**, 10–43 (1953); MR **14**, 989], but many fundamental questions had been left open. The author answers a number of these questions and suggests some new problems. He proves in quite an elegant way that every separable normed linear space is scm, and supplies a table giving the status, relative to the properties sc, sm, and scm, of various non-separable analogues of the Banach spaces (c_0), (l_1), and (m). For example, the space (m) of all bounded real sequences is sc but not sm, while the space of all bounded real functions on an uncountable set is neither sc nor sm. This example of a space which is not sc solves a problem of long standing, stated explicitly by Dixmier [Duke Math. J. **15**, 1057–1071 (1948); MR **10**, 306; the same example had been mentioned without proof by R. S. Phillips, Amer. J. Math. **65**, 108–136 (1943); MR **4**, 218]. Several other specific spaces are discussed, and the paper ends with a result showing the impossibility of a certain generalization of a theorem of Kakutani and Dugundji on simultaneous extension of continuous functions [Dugundji, Pacific J. Math. **1**, 353–367 (1951); MR **13**, 373]. The following are among the unsolved problems mentioned: Must a reflexive space be sc? Is there a space which is sm but not sc? Is there a nonreflexive nonseparable scm space? Is there an sm space whose dual is not sc?

V. L. Klee, Jr. (Seattle, Wash.).

Marinescu, G. Sur la différentielle et la dérivée dans les espaces normés. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 213–219 (1954). (Romanian. Russian and French summaries)

This note gives sufficient conditions that a Gateaux differential of a function from one normed space to another be a Fréchet differential, and also conditions necessary and sufficient that a function be a derivative in the neighborhood of a point. *M. M. Day* (Urbana, Ill.).

MacDowell, Robert. Banach spaces and algebras of continuous functions. Proc. Amer. Math. Soc. **6**, 67–78 (1955).

Let X be a compact Hausdorff space, let $C(X)$ be the space of all real continuous functions on X. The author studies the structure of $C(X)$ when X is non-trivially a product space, and when X contains as subspace a product space with non-empty interior. Characterizations of $C(X)$ are obtained, both as a Banach space and as a Banach algebra. For example, X is a product space if and only if $C(X)$ contains two subalgebras F and G, each containing the constant functions, with the property that if $f \in F$, $g \in G$, then either $\|f+g\|=\|f\|+\|g\|$ or $\|f-g\|=\|f\|+\|g\|$, and such that F and G together generate a subalgebra dense in $C(X)$. *S. B. Myers* (Ann Arbor, Mich.).

MacNerney, J. S. Stieltjes integrals in linear spaces. Ann. of Math. (2) **61**, 354–367 (1955).

Let S be a Banach space and let B be the Banach algebra of bounded linear operators from S to S. For functions F, G, H, from real numbers into B the Stieltjes integral $\int_a^b F\,dG,\,H$ is defined when the limit of the appropriate sums exists in the strong topology of B. A B-valued function M of two real variables is called harmonic [terminology of Wall, Arch. Math. **5**, 160–167 (1954); MR **15**, 801] if for each value of one variable it is continuous and of bounded variation for each finite interval of the other. It is shown that some of Wall's results carry to this general setting; in particular, there is a one-to-one correspondence between H, the class of harmonic functions, and ϕ, the class of functions F from real numbers into B such that $F(0)=0$ and F is of bounded variation on each interval, under which, for all s and t, $M(s,t)=1+\int_s^t dF(u)M(u,t)$. M can also be represented as a product integral $M(s,t)=\prod_s^t (1+dF)$. There are applications of these functions to a non-linear integral equation and to continued fractions. *M. M. Day.*

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. Note on Banach spaces. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **4**, 319–323 (1954).

Although the authors speak about "regularity" of functions with domain and range in a complex Banach space, there is really nothing about analyticity in the paper, the only important feature being some kind of differentiability. The space may equally well be real. The definition of differentiability as given is not entirely clear to the reviewer, but the main result is certainly valid if differentials are taken in the sense of Fréchet. Suppose that f is continuously differentiable in the convex set K of the Banach space L. Let f' be the Fréchet derivative of f (so that $f'(x)$ is a bounded linear operator on L). Suppose that there exists a bounded linear operator g on L such that $\|gf'(x)-I\|<1$ when $x \in K$. Then $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ in K. The proof uses only the most elementary inequalities and an integral identity involving f and f'. *A. E. Taylor* (Los Angeles, Calif.).

Ringrose, J. R. Compact linear operators of Volterra type. Proc. Cambridge Philos. Soc. **51**, 44–55 (1955).

The author gives conditions for a compact linear operator A defined on a Banach space to be quasi-nilpotent. He shows that if there is a resolution of the identity $\{E_\lambda\}$, where $E_0=0$, $E_1=I$ which is strongly continuous and uniformly bounded such that $AE_\lambda=E_\lambda AE_\lambda$ $(0\leqq\lambda\leqq1)$, then A is quasi-nilpotent. Integral operators of the Volterra type on $L^p(a,b)$, $1\leqq p<\infty$, are shown to be quasi-nilpotent with the use of this result. Applications to the existence and uniqueness of solutions of integral equations are given.

B. Yood (Eugene, Ore.).

Neveu, Jacques. Sur une hypothèse de Feller à propos de l'équation de Kolmogoroff. C. R. Acad. Sci. Paris **240**, 590–591 (1955).

Let C be the Banach space of continuous functions (uniform topology) on the interval $[r_1, r_2]$. The semigroup

$\{T_t, t>0\}$ on this space is strongly continuous for $t>0$, takes the class of non-negative functions into itself, the function 1 into itself, and $(T_h f)(x)=o(h)$ if f vanishes in a neighborhood of x. Let Ω be the infinitesimal operator of the semigroup. The author displays explicitly the general form of $(\Omega f)(x)$ at each point x of $[r_1, r_2]$, a somewhat generalized second-order differential operator, depending on a classification of x as regular or irregular. If the functions x and x^2 are in the domain of Ω, Ω becomes an ordinary second-order operator. [See Feller, Ann. of Math. (2) 60, 417–436 (1954) [MR 16, 488] for similar results from a different point of view.] *J. L. Doob* (Urbana, Ill.).

Kendall, David G. Bernstein polynomials and semigroups of operators. Math. Scand. 2, 185–186 (1954).

Let $[T(t); t\geqq 0]$ be a semi-group of linear bounded operators on a real Banach space $\mathfrak{X}$ to itself such that $T(0)=I$ and $\lim_{t\to 0+}\|T(t)x-x\|=0$ for each $x\in\mathfrak{X}$. The author proves that in this case

$$\lim_{n\to\infty}\|\{(1-t)I-tT(1/n)\}x-T(t)x\|=0$$

for each $x\in\mathfrak{X}$ whenever $0\leqq t\leqq 1$, the convergence being uniform in this interval. He then goes on to show that the usual Bernstein approximation theorem for continuous functions can be obtained as a special instance of the above result. *R. S. Phillips* (Stanford, Calif.).

Berge, Claude. Sur une convexité régulière non linéaire et ses applications à la théorie des jeux. Bull. Soc. Math. France 82, 301–315 (1954).

In the first of the two parts of this paper, the author establishes results which are generalizations of the classical Banach space results on regularly convex sets: Let $\mathfrak{F}$ be a conically convex (f_1, f_2, in $\mathfrak{F}$, $\lambda_1, \lambda_2\geqq 0 \Rightarrow \lambda_1 f_1+\lambda_2 f_2$ in $\mathfrak{F}$) set of convex lower semicontinuous functionals on a linear topological space E. $X\subset E$ is called $\mathfrak{F}$-regularly convex if for each x_0 not in X there is an f in $\mathfrak{F}$ and a real α for which $f(x_0)>\alpha$, $f(x)<\alpha$ (x in X). Analogously, an $\mathfrak{F}$-hyperplane $H_f(\alpha)$ is defined to be $\{x\,|\,f(x)=\alpha\}$, where f is in $\mathfrak{F}$. Theorem: If C is compact convex and if C' is $\mathfrak{F}$-regularly convex (hence convex closed) and if $C\cap C'$ is empty, then there is an f in $\mathfrak{F}$ such that $f(x)\geqq\alpha$, x in C, $f(x)<\alpha$, x in C', α real, i.e., there is an $H_f(\alpha)$ strictly separating C and C'. Similar extensions of theorems on extreme points and supporting hyperplanes, etc., are proved.

In the second part of the work, the first set of results is applied to extensions of game theory, i.e., to the establishment of a value and to the discussion of the sets of good strategies for each player in games for which the payoff function is of the form $\Phi(p, q)=\sum_{j=1}^{n}\sum_{i=1}^{m} p_i\varphi_{ij}(q_j)$ and φ_{ij} are convex, continuous in $[0, 1]$ and $\varphi_{ij}(0)=0$. For example, the set of good strategies for either player, say the first, in the standard or reversed game is a nonempty convex compact set in the space $\mathcal{P}_m$ of mixed strategy vectors for the first player and the set of good strategies is also $\mathfrak{F}_v$-regularly convex where $\mathfrak{F}_v$ is the set of functions

$$\Phi'(\xi, y)=\sum_{i,j}\xi_i[\varphi_{ij}(y_j)-\nu y_j],\quad \xi_i\geqq 0,\quad \sup_p\inf_q\Phi(p, q)=\nu.$$

B. Gelbaum (Minneapolis, Minn.).

Davis, Philip. Linear functional equations and interpolation series. Pacific J. Math. 4, 503–532 (1954).

Let B be a $2n$-dimensional region in the $z=(z_1, \cdots, z_n)$-space of n complex variables. The space of regular single-valued analytic functions f in B for which $\|f\|^2=\int_B|f|^2d\omega<\infty$ is called $L^2(B)$. Let L be a linear operator defined for $f\subset L^2(B)$ and such that $L(f)$ is a regular analytic function in B. The paper is concerned with conditions for the solvability of equations of the form

(1a) $\quad L(u)=0,\qquad$ (1b) $\quad L(u)=f,$

with the generation of a complete set of solutions and with the representation of such solutions. The homogeneous equation (1a) is treated first. First of all, it is shown that it has a non-trivial solution in $L^2(B)$ if and only if the set of linear functionals $\bar{L}_n=L_n(L)$ is "incomplete" (i.e., not total in the sense of Banach) for $L^2(B)$ where $\{L_n\}$ is a complete (total) sequence of bounded linear functionals defined over the set of functions regular in B. Moreover, a representation of a solution $g(Z)$ of (1a) is given in the following form:

$$(2)\qquad g(Z)=f(Z)-\int_B K_I(Z, \overline{W})f(W)d\omega_W\qquad (f\subset L^2(B)),$$

and conversely every g of the form (2) is proved to be a solution of (1a). Here

$$K_I(Z, \overline{W})=\sum_{k=0}^{\infty}\phi_k^*(Z)\overline{\phi_k^*(W)},$$

where the $\phi_k^*(Z)$ may be obtained as follows: let $K_B(Z, \overline{W})$ be the Bergmann kernel function of B, denote the $\bar{L}_n$ operating on $\overline{W}$ by $\bar{L}_{n,\overline{W}}$, and set $\phi_n(Z)=L_{n,\overline{W}}(K_B(Z, \overline{W}))$. The set $\{\phi_n^*\}$ is then obtained from the set $\{\phi_n\}$ by the Gram-Schmidt orthogonalization procedure. By a "double orthogonalization procedure" [J. L. Walsh and P. Davis, J. Analyse Math. 2, 1–28 (1952); MR 16, 580] a system of functionals $\{\bar{L}^*\}$ is constructed which is "equivalent" to the system $\{\bar{L}_n\}$ and bi-orthogonal to the ϕ_m^*, i.e., $L_n^*(\phi_m^*)=\delta_{nm}$. An alternate representation of the solutions (2) of (1a) is then given in the form

$$(3)\qquad g(Z)=f(Z)-\sum_{k=0}^{\infty}L_k^*(f)\phi_k^*(Z).$$

If $\theta_0, \theta_1, \cdots$ is a complete set of functions in $L^2(B)$, then the set obtained by replacing f by θ_n in (2) or (3) is a complete set of solutions of (1a).

In case of the inhomogeneous equation (1b) it is first shown that it is equivalent to the system $\bar{L}_k(u)=\beta_k$, where $\beta_k=\bar{L}_k(f)$. Let now $\{L_k^*\}$ and $\{\phi_k^*\}$ have the same meaning as above. By the definition of the L_k^* we have $L_k^*=\sum_{p=0}^{k}\alpha_{kp}L_p$ with constant α_{kp}. It is then proved that (1b) has a solution in $L^2(B)$ if and only if

$$(4)\qquad \sum_{k=0}^{\infty}\left|\sum_{p=0}^{\infty}\alpha_{kp}\beta_p\right|^2<\infty$$

and that the solution is unique in $L^2(B)$ if and only if the set $\bar{L}_k$ is complete. Moreover, if (4) holds, the series

$$\sum_{k=0}^{\infty}\left(\sum_{p=0}^{k}\alpha_{kp}\beta_p\right)\phi_k^*(Z)$$

converges absolutely and uniformly in every closed subset of B and represents a solution of (1b). If to (1a) or (1b) auxiliary conditions are added of the form $A_k(u)=\alpha_k$, where the A_k are bounded linear functionals and the α_k are constants which in case of (1a) are all zero, the above methods and results still hold with very minor changes.

In the remaining part of the paper the convergence of the series (3) is investigated in cases where f is not necessarily in $L^2(B)$, and the connection with stability questions is discussed. Finally, the methods of the paper are extended to systems of equations. *E. H. Rothe* (Ann Arbor, Mich.).

Deny, Jacques, et Lions, Jacques Louis. Espaces de Beppo Levi et applications. C. R. Acad. Sci. Paris **239**, 1174–1177 (1954).

The Beppo Levi space referred to in the title is the space of functions with square integrable first derivatives (finite Dirichlet integral) over some open set of n-dimensional space. The starting point of the authors is the familiar decomposition of such a function by orthogonal projection into the sum of a harmonic function and one that vanishes on the boundary in a generalized sense, i.e. is the limit in the Dirichlet norm of smooth functions vanishing in a boundary strip [see, e.g., Courant and Hilbert, Methoden der mathematischen Physik, Bd. II, Springer, Berlin, 1937, chap. VII]. Deny [Acta Math. **82**, 107–183 (1950); MR **12**, 98] has shown that every function with finite Dirichlet integral is, after alteration on a set of measure zero, precise, i.e. is continuous outside of sets of arbitrary small capacity. The main result of this paper is an intrinsic characterization of such precise functions which vanish on the boundary in a generalized sense: they have quasi limit zero at all boundary points outside of a set fo exterior capacity zero; here $n\geqq 3$. The authors also show that such a function can be represented by the familiar formula involving Green's function. In section 3 some observations are made about the Neumann problem, operating with the class of functions which have zero normal derivative on the boundary in the weak sense. The key here is Poincaré's inequality (Courant-Hilbert, loc. cit.); the authors give a number of equivalent conditions for its validity. *P. D. Lax.*

Tillmann, Heinz-Günther. Dualität in der Potentialtheorie. Portugal. Math. **13**, 55–86 (1954).

This paper presents a duality theory for linear spaces of harmonic functions which is quite similar to the duality theory for linear spaces of analytic functions, as developed by C. da Silva Dias [Thesis, São Paulo, 1951; MR **13**, 249], G. Köthe [J. Reine Angew. Math. **191**, 30–49 (1953); MR **15**, 132], and extended by A. Grothendieck [ibid. **192**, 35–64, 77–95 (1953); MR **15**, 438, 963]. The basic tool here, corresponding to Cauchy's integral formula for analytic functions, is Green's third integral formula, which permits the representation of the values of a harmonic function in terms of a surface integral. Let R^m be m-dimensional real Euclidean space ($m\geqq 3$) and let $\hat{R}^m$ be the compactification of R^m by the addition of a point at infinity. If M is a point-set in $\hat{R}^m$, $\mathfrak{P}(M)$ is the linear space of functions, each harmonic on a neighborhood of M, with identification of functions which coincide on such a neighborhood. To define a topology on $\mathfrak{P}(M)$ three cases are considered, according as M is open, compact, or neither. The dual of $\mathfrak{P}(M)$ is identified with $\mathfrak{P}(M')$, where $M'=\hat{R}^m-M$. If u is in the dual of $\mathfrak{P}(M)$, the corresponding element $\tilde{u}$ of $\mathfrak{P}(M')$ is found by applying u to the harmonic function $|x-\xi|^{2-m}$, where ξ is a fixed point in M'. The values of u are then represented by a surface integral involving $\tilde{u}$. Another topology for $\mathfrak{P}(M)$ is defined, which turns out to be the same as the strong topology of $\mathfrak{P}(M)$ when the latter is regarded as the dual of $\mathfrak{P}(M')$. The strong topology of $\mathfrak{P}(M')$ as the dual of $\mathfrak{P}(M)$, and its topology as a space of harmonic functions, are the same in case M is either open or closed. Other theorems about duality relations are given, and there are theorems about the representation of continuous linear mappings of $\mathfrak{P}(M)$ into a locally convex linear space E, in terms of vector-valued harmonic functions.

A. E. Taylor (Los Angeles, Calif.).

Arbault, Jean. Sur les transformations de Reynolds quasi régulières. C. R. Acad. Sci. Paris **239**, 949–951 (1954).

Further analysis of the concept of a "quasi-regular" Reynolds operator (on a space of functions from a set E to an ordered integral domain), introduced by the author previously [same C. R. **239**, 858–860 (1954); MR **16**, 145].

G. Birkhoff (Cambridge, Mass.).

Kampé de Fériet, Joseph. Construction des transformations de Reynolds régulières. C. R. Acad. Sci. Paris **239**, 934–936 (1954).

Let M be the space of all real, non-negative functions on a set X, which are measurable with respect to a σ-field F of subsets of X. Let T be any "regular" Reynolds operator on M, in the sense of M. L. Dubreil-Jacotin [same C. R. **236**, 1136–1138 (1953); MR **14**, 839]. Then it is proved that $Tf=\int f(y)\,d\lambda_x$, for a suitably defined measure function defined on F. *G. Birkhoff* (Cambridge, Mass.).

Raman, P. K. On a class of linear spaces in function theory. J. Indian Math. Soc. (N.S.) **18**, 127–130 (1954).

Let π_r denote the linear space of all power series $x(\omega)=\sum_{n=1}^{\infty} x_n\omega^{n-1}$, with radius of convergence at least $r>0$, in the topology of uniform convergence on compact sets. If $x\in\pi_r$, let $(x)_i=\sum_{n=1}^{\infty} x_n{}^i\omega^{n-1}$, and let $L[x]$ denote the closed linear space generated by the $(x)_i$ for $i=1, 2, \cdots$. If $r>1$, a necessary and sufficient condition that $L[x]=\pi_r$ is that all the x_n are distinct and non-zero. (Previously, V. Ganapathy Iyer [J. Indian Math. Soc. (N.S.) **17**, 183–185 (1954); MR **15**, 719] had shown the sufficiency of this condition when $r=\infty$.) While this condition is necessary in order that $L[x]=\pi_1$, an example is given to show that it is not sufficient in this case. *M. Henriksen.*

Krasnosel'skiĭ, M. A., and Sobolev, V. I. Conditions of separability of Orlicz spaces. Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 59–68 (1955). (Russian)

This paper has two principal results: (1) The establishment of an isometry between $L_M{}^*(G)$ and $L_M{}^*(I)$, where I is the closed interval $[-\frac{1}{2}m(G), \frac{1}{2}m(G)]$, G is a closed set in Euclidean space and m is Lebesgue measure. (2) Theorem: $L_M{}^*$ is separable if and only if M satisfies the Δ_2 condition for large u: $M(2u)\leqq kM(u)$ for $u\geqq 1$, k a constant. [For notations and other definitions see two previous papers by the first author and Rutickiĭ [Dokl. Akad. Nauk SSSR (N.S.) **81**, 497–500 (1951); **89**, 601–604 (1953); MR **13**, 357; **15**, 137]. In the first of these references the relevance of the present results to the study of nonlinear integral operators is indicated without proof.] Proofs in the paper under review are simple and elegant. *B. R. Gelbaum.*

Horváth, J. I. An asymptotical method for the calculation of eigenfunctions. Acta Phys. Acad. Sci. Hungar. **4**, 183–186 (1954).

This paper modifies trivially an earlier paper by Biedenharn and Blatt [Phys. Rev. (2) **93**, 230–232 (1954); MR **15**, 745], which presented a variant of the standard first-order perturbation formulae for a self-adjoint operator H on Hilbert space by replacing the eigenvectors of an approximation H_0 to H by an arbitrary complete orthonormal set of trial vectors. This paper contains a mistake (possibly typographical) in assuming unnecessarily that H and Q commute. This would require

$$H\Phi_k=H(\psi_k+Q\psi_k)=(I+Q)H\psi_k=\lambda_k\Phi_k;$$

hence the trial vectors Φ_k would have to coincide with the

exact eigenvectors ψ_k of H in the nondegenerate case, which would be trivial. *F. H. Brownell* (Seattle, Wash.).

Maak, Wilhelm. Periodizitätseigenschaften unitärer Gruppen in Hilberträumen. Math. Scand. 2, 334–344 (1954).

The author uses the Alaoglu-Birkhoff proof of the ergodic theorem to prove the existence of mean values for certain functions defined on an arbitrary group G of unitary operators on Hilbert space H. Among these functions are the functions $(h, xf)^2$, where h and $f \in H$ and $x \in G$. Call a vector f in H a furtive vector if $M_x(h, xf)^2=0$ for every h in H. Then H is decomposed into the closed linear manifold of furtive vectors and the orthogonal manifold of almost periodic (under G) vectors. (K. Jacobs, in the paper reviewed below, has extended this last result to bounded groups in uniformly convex (uc) spaces with uc conjugate spaces.)

M. M. Day (Urbana, Ill.).

Jacobs, Konrad. Periodizitätseigenschaften beschränkter Gruppen im Hilbertschen Raum. Math. Z. 61, 408–428 (1955).

Let G be a bounded group of linear transformations of a Banach space B. Call an element h of B an almost periodic vector if the orbit of h, $\{gh \mid g \in G\}$, is totally bounded (in norm): call f a furtive vector if there is a sequence $\{g_n\}$ of elements of G such that $\{g_nf\}$ tends weakly to zero. Using the techniques of his earlier note [Math. Ann. 128, 340–349 (1954); MR 16, 374] the author extends a theorem proved in the paper reviewed above for Hilbert space. Let P and F be the sets of almost periodic and of furtive vectors in B. Then if B and B^* are isomorphic to uniformly convex spaces, P and F are closed, linear, invariant subspaces of B which intersect in the 0 element of B and which span B; moreover, the projection of B on P along F is a bounded linear projection. In order to prove that this is a generalization of Maak's result, it is shown that the definition of furtive vector above and Maak's definition describe the same subset of B, in case B is Hilbert space.

M. M. Day (Urbana, Ill.).

Kadison, Richard V. On the general linear group of infinite factors. Duke Math. J. 22, 119–122 (1955).

This paper is a continuation of a previous paper [Trans. Amer. Math. Soc. 76, 66–91 (1954); MR 15, 721] where the author considered the problem of determining all uniformly closed subgroups of the group M_g of all invertible operators in a Murray-von Neumann factor M. All cases except factors of type II_∞ were disposed of in the first paper. Here it is proved that the result for type II_∞ is identical with that for type I_∞ [see the review of the first paper]. In fact, this paper contains a new proof which covers both types I_∞ and II_∞. *C. E. Rickart* (New Haven, Conn.).

Yood, Bertram. Periodic mappings on Banach algebras. Amer. J. Math. 77, 17–28 (1955).

Let T be either an automorphism or anti-automorphism of a real Banach algebra B. The author studies continuity properties of such T which have a finite period n. In this case it is easy to see that B is also a Banach algebra under the new norm $\|x\|_1=\|Tx\|$, so T will be continuous provided the norms $\|x\|$ and $\|x\|_1$ are equivalent. Now, for any two norms $\|x\|$ and $\|x\|_1$ on a Banach algebra B, denote by S the set of elements x such that there exists $\{x_n\}$ for which $\|x_n-x\|\to0$ and $\|x_n\|_1\to0$. Then $S=(0)$ is necessary and sufficient that $\|x\|$ and $\|x\|_1$ be equivalent. It was proved by the reviewer [Ann. of Math. (2) 51, 615–628 (1950); MR 11, 670] that S is a two-sided ideal which is closed and consists entirely of topological divisors of zero with respect to both norms. The following additional properties of the "separating ideal" S are obtained here: (1) If S_1 is the separating ideal for the two norms on the algebra S, then $S^2\subseteq S_1$; hence if the norms are equivalent on S, then S is a zero algebra. (2) S_1 is contained in no proper regular (one- or two-sided) ideal of S. (3) Either the norms are equivalent on B or S is a radical algebra in the sense of Brown-McCoy [Amer. J. Math. 69, 46–58 (1947); MR 8, 433]. Returning to the case of a periodic T, the associated S is called the separating ideal for T. Define $P_n=n^{-1}(I+T+T^2+\cdots+T^{n-1})$, where n is the period of T. Then $P_n^2=P_n$ and $B=H\oplus K$ where $H=\{x \mid Tx=x\}$ and $K=\{x \mid P_nx=0\}$. It is proved that T is continuous if, and only if, H and K are closed and T is continuous on K. Furthermore $S\subseteq\bar{K}$. One of the main results obtained here is the following: Let B be semi-simple in the sense of Jacobson [ibid. 67, 300–320 (1945); MR 7, 2]. Then any T with period 2^n such that $S\subset K$ (e.g., if K is closed) is continuous.

[Note: The author reports an error in the proof of his Lemma 4.1 (2) which apparently holds in general only for period two. This requires adding the hypothesis $T(S)=S$ to the results 4.1 (3, 4), 4.2(a), 4.3, 4.4 (except $n=2$) and 4.10. Lemma 4.1(2) was also used in the proof of Theorem 4.6 (the last result quoted above); However, the author has supplied another proof independent of 4.1(2).]

C. E. Rickart (New Haven, Conn.).

Lumer, G. Fine structure and continuity of spectra in Banach algebras. An. Acad. Brasil. Ci. 26, 229–233 (1954).

The author discusses some questions concerning singular elements in Banach algebras. No proofs are given. The following are among the results stated: Let B be a Banach algebra with unit e, let S be the set of singular elements in B. The following subsets of S are introduced.

$$\begin{gathered}S_+=\{x: x \text{ is not right invertible}\},\\ S_-=\{x: x \text{ is not left invertible}\},\\ D_+=\{x: \exists y_n, \|y_n\|=1, y_nx\to0\},\\ D_-=\{x: \exists y_n, \|y_n\|=1, xy_n\to0\},\\ D_{0+}=\{x: \exists y\neq0, yx=0\},\\ D_{0-}=\{x: \exists y\neq0, xy=0\},\\ D=D_+\cup D_-,\quad C_+=S_+\cup D_+',\\ C_-=S_-\cup D_-',\quad C=C_+\cup C_-.\end{gathered}$$

For K one of these sets and x in B, $K(x)=$ the set of complex numbers λ with $\lambda e-x$ belonging to K. The map x into $K(x)$ is called upper semicontinuous at x_0 if for any open set $U\supset K(x_0)$ there exists $\delta>0$ with $U\supset K(x)$ if $\|x-x_0\|<\delta$. Then $S(x)$, $S_+(x)$, $S_-(x)$, $D(x)$, $D_+(x)$, $D_-(x)$ are upper semi-continuous. If the set X of all compact subsets of the complex plane is given the Hausdorff metric, it is known that $S(x)$ is not continuous as map from B to X. The author states that if, for some x_0, $S(x_0)$ is the union of a totally disconnected set and a set contained in the closure of $C(x_0)$, then $S(x)$ is continuous at x_0. The author also mentions extensions to the case where the algebra lacks a unit.

J. Wermer (Providence, R. I.).

Hartman, S. Quelques remarques sur les expansions de Fourier. Studia Math. 14 (1954), 200–208 (1955).

This paper consists of corollaries of theorems of Gel'fand [Mat. Sb. N.S. 9(51), 51–66 (1941); MR 3, 51] and Beurling [Nionde Skandinaviska Matematikerkongressen, Helsing-

fors, 1938, Mercator, Helsingfors, 1939, pp. 345–366]. Let R be a complex Banach algebra consisting of sequences $a=\{a_n\}_{n=1}^{\infty}$ of complex numbers, where all operations are termwise, the norm satisfies the usual inequality

$$\|ab\| \leqq \|a\| \cdot \|b\|,$$

and the set E consisting of all sequences $\{a_n\}$ vanishing except for a finite number of co-ordinates is contained in and is dense in R. If R has no unit, adjoin a unit e by the usual method, obtaining the Banach algebra R^* consisting of all elements $\lambda e+a$, where λ is complex and $a \in R$. The following elementary theorem is proved. Let h be a multiplicative linear functional on R^* different from zero. Then either $h(\lambda e+a)=\lambda$ or there exists a positive integer j such that $h(\lambda e+a)=a_j$. [Reviewer's note. An obvious analogue of this theorem holds for Banach algebras of continuous functions on locally compact Hausdorff spaces, where E is replaced by the set of all continuous functions vanishing outside of compact sets.] A number of applications are made of this theorem, of which the following is typical. Let $\{a_n\}_{n=1}^{\infty}$ by the sequence of non-zero Fourier coefficients of a Besicovitch (Bohr) almost periodic function on $(-\infty, \infty)$. If λ is a complex number such that $\lambda+a_n\neq 0$, for all n, then $\{a_n(\lambda+a_n)^{-1}\}_{n=1}^{\infty}$ is the sequence of non-zero Fourier coefficients of a Besicovitch (Bohr) almost periodic function.

E. Hewitt (Seattle, Wash.).

Hörmander, Lars. La transformation de Legendre et le théorème de Paley-Wiener. C. R. Acad. Sci. Paris **240**, 392–395 (1955).

Soit $z=x+iy \in C^n$, x, $y \in R^n$; on considère l'espace dual de C^n, de point générique $\zeta=\xi+i\eta$, ξ, $\eta \in R^n$. Soit $z\rightarrow u(z)$ une fonction donnée sur C^n, à valeurs réelles, finies ou non. On pose:

$$u'(\zeta)=\inf_y \sup_x \,(u(x+iy)-\langle x,\eta\rangle-\langle y,\xi\rangle),$$

et

$$\tilde{u}'(z)=\inf_\eta \sup_\xi \,(u'(\xi+i\eta)+\langle x,\eta\rangle+\langle y,\xi\rangle).$$

On suppose que la fonction u vérifie $\tilde{u}'=u$. L'auteur donne d'abord la condition nécessaire et suffisante pour qu'il en soit ainsi (et même dans une situation plus générale). Posons ensuite: $W(y,\eta)=\sup_x (u(x+iy)-\langle x,\eta\rangle)$. On désigne par $\mathcal{S}_u$ l'espace des fonctions $z\rightarrow f(z)$ définies sur l'ensemble $\{z \in C^n \mid u(z)<\infty\}$, et qui possèdent les propriétés suivantes: (1) $t\rightarrow f(x+i(y_1+t(y_2-y_1)))$ est analytique dans la bande $0<\operatorname{Re} t<1$, si $W(y_i,\eta)$, $i=1,2$, pour au moins un η; (2) $x\rightarrow f(x+iy)$ est indéfiniment différentiable dans son domaine d'existence; (3) quels que soient les polynomes P et Q, il existe une constante C telle que

$$|P(z)Q(\partial/\partial x)f(z)| \leqq C \exp u(x+iy)$$

(cette condition permet de définir de façon naturelle les voisinages de 0 dans $\mathcal{S}_u$).

Sur l'ensemble $\{\zeta \in C^n \mid u'(\zeta)<\infty\}$, on définit la fonction

$$\zeta\rightarrow\hat{f}(\zeta)=\int f(z) \exp i\langle z,\zeta\rangle dx,$$

y fixé avec $W(y,\eta)<\infty$ (de telle sorte que l'intégrale converge absolument). Cette fonction est indépendante de y; l'auteur l'appelle "transformée de Laplace" de f. On définit de façon analogue à précédemment un espace $\mathcal{S}_{u'}$. Dans ces conditions: Théorème: l'application $f\rightarrow\hat{f}$ est un isomorphisme de $\mathcal{S}_u$ sur $\mathcal{S}_{u'}$.

Un exemple trivial est le suivant: $u(z)=+\infty$ si $y\neq 0$, $=0$ si $y=0$. Alors $\mathcal{S}_u=\mathcal{S}$, espace des fonctions à décroissance rapide sur R^n, espace introduit par L. Schwartz [Théorie des distributions, t. II, Hermann, Paris, 1951; MR 12, 833]. La fonction f est alors la transformée de Fourier de f, et le théorème est conséquence immédiate de la formule d'inversion de Fourier. C'est sur ce résultat que Schwartz (loc. cit.) a construit la théorie de la transformation de Fourier des distributions tempérées (par transposition). La même méthode s'applique ici et permet de définir la transformée de Laplace de fonctionnelles (qui ne sont pas en général des distributions). Gel'fand et Šilov [Uspehi Mat. Nauk (N.S.) **8**, no. 6(58), 3–54 (1953); MR **15**, 867] ont donné un certain nombre de théorèmes d'isomorphisme analogues pour des classes spéciales de fonctions; l'auteur annonce que son théorème contient notamment comme cas particuliers les résultats de Gelfand et Šilov.

J. L. Lions (Nancy).

Gurevič, B. L. New types of spaces of fundamental and generalized functions and Cauchy's problem for systems of finite-difference equations. Dokl. Akad. Nauk SSSR (N.S.) **99**, 893–895 (1954). (Russian)

Soit $\mu(x)$ une fonction continue pour $x\geqq 0$, monotone, avec $\mu(0)=0$, $\mu(\infty)=\infty$; soit ω la fonction inverse de μ; posons $M(x)=\int_0^x \mu(\xi)\,d\xi$, $\Omega(\tau)=\int_0^\tau \omega(\theta)\,d\theta$. On désigne par K_M l'espace des fonctions φ définies sur R^N, à valeurs complexes, telles que pour tout p, il existe deux constantes C et C_1 telles que

$$|D^p\varphi(x)| \leqq C_1 \exp(-M(C|x|)), \text{ pour tout } x \in R^N.$$

On désigne par Z^Ω l'espace des fonctions ψ définies sur R^N à valeurs complexes, prolongeables analytiquement à R^N+iR^N, telles que, pour tout polynome P, il existe deux constantes A et A_1 telles que, pour tout $\tau \in R^N$ $(s=\sigma+i\tau)$, on ait

$$\int_{R^N} |P(s)|^2 |\psi(\sigma+i\tau)|^2 d\sigma \leqq A_1 \exp(\Omega(A|\tau|)).$$

L'auteur annonce le résultat suivant: la transformation de Fourier applique biunivoquement K_M sur Z^Ω.

Des résultats de ce genre (mais plus particuliers) ont été donnés récemment par Gel'fand et Šilov [Uspehi Mat. Nauk (N.S.) 8, no. 6(58), 3–54 (1953); MR **15**, 867], obtenus eux mêmes comme généralisation du procédé de Schwartz pour définir la transformation de Fourier des distributions tempérées [Théorie des distributions, t. II, Hermann, Paris, 1951; MR **12**, 833]. [Note du reviewer: il faut également citer la note de L. Hörmander analysée ci-dessus.]

L'auteur applique ensuite ce théorème à certains problèmes de Cauchy. On introduit l'espace K_M' dual de K_M, et Z'^Ω dual de Z^Ω, puis $K_M'^m$ produit de m espaces égaux à K_M'; de même $(Z'^\Omega)^m$ (L'auteur note ces espaces $T^{(m)}(K_M)$, $T^{(m)}(Z^\Omega)$, et d'ailleurs ne précise pas la topologie sur K_M et Z^Ω). Par transposition, la transformation de Fourier (généralisée) établit un isomorphisme entre ces espaces. On cherche $t\rightarrow v(t)$, application une fois continûment dérivable de $t\geqq 0$ dans $(Z'^\Omega)^m$, (noter que Z'^Ω n'est pas un espace de distributions) solution de

$$\frac{d}{dt}v(t)=F(t)v(t), \tag{1}$$

où $t\rightarrow F(t)$ est une application continue de $t\geqq 0$ dans l'espace des multiplicateurs de $(Z'^\Omega)^m$ (analogue à l'espace O_M de Schwartz), avec la condition initiale

$$v(0)=v_0 \in (Z'^\Omega)^m. \tag{2}$$

Au système (1) on associe le système différentiel ordinaire $dv(s,t)/dt=F(s,t)v(s,t)$, $v(s,t)=$ valeur de $v(t)$ au point s de R^N+iR^N, etc. Si la matrice résolvante de ce système est un multiplicateur sur $(Z'^{\Omega})^m$, alors le problème (1) (2) admet une solution unique (dans l'inégalité (2) de l'auteur, il faut lire: $\cdots t)dt|\leqq A_1e\cdots$). Par transformation de Fourier inverse, on en déduit aussitôt un théorème d'existence et d'unicité pour le problème de Cauchy transformé, à valeurs dans l'espace de vecteurs distributions $K_M'^m$. [Note du reviewer: cette méthode a fait l'objet d'un article de L. Schwartz, Ann. Inst. Fourier, Grenoble **2**, 19–49 (1951); MR **13**, 242.] Application à un problème aux différences finies. Pour terminer, l'auteur donne, par application d'une méthode due à Kostyučenko et Šilov [Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 141–148 (1954); MR **16**, 253] une condition suffisante portant sur v_0 pour que la solution du problème de Cauchy soit une fonction usuelle.

J. L. Lions (Nancy).

Cristescu, Romulus. Introduction to the theory of partially ordered spaces. Gaz. Mat. Fiz. Ser. A. **6**, 527–537 (1954). (Romanian)

Calculus of Variations

Fleming, W. H., and Young, L. C. A generalized notion of boundary. Trans. Amer. Math. Soc. **76**, 457–484 (1954).

This paper treats a number of general minimum problems in the calculus of variations, making use of the linear-space approach developed by Young in various papers [e.g., Bull. Soc. Math. France **79**, 59–84 (1951); MR **13**, 731]. Their results are applied, specifically, to generalize a minimum theorem of Cesari [Amer. J. Math. **74**, 265–295 (1952); MR **14**, 292], Sigalov [Uspehi Mat. Nauk (N.S.) **6**, 16–101 (1951) = Amer. Math. Soc. Transl. no. **83** (1953); MR **13**, 257; **14**, 769], and Danskin [Riv. Mat. Univ. Parma **3**, 43–63 (1952); MR **14**, 292] and to extend theorems of Kreĭn and Mil'man on extreme points.

A vector-valued mapping $x(u,v)$ from the unit square to m-space R^m is called a Dirichlet representation if $x(u,v)$ is absolutely continuous in the sense of Tonelli and if $\iint(x_u^2+x_v^2)\,dudv<\infty$. A Dirichlet surface with Dirichlet representation $x(u,v)$ is the non-negative linear functional L defined over the space E^m of positively homogeneous m-dimensional calculus-of-variations integrands f by means of the following integral:

$$(L,f)=\iint f[x(u,v),J(u,v)]\,dudv.$$

A generalized surface, as earlier defined by Young, is an arbitrary non-negative linear functional L over E^m. Convergence is weak *, i.e. $L\to L_0$ if $(L,f)\to(L_0,f)$ for all f. An exact integrand ϕ in E^m is one for which $(L,\phi)=0$ whenever L is a closed polyhedron. The generalized boundary (abbreviated g-boundary) of a generalized surface L is the linear functional $\lambda(L)$ obtained by restricting the domain of L to exact integrands. It amounts to a generalization of line integration. A g-boundary λ is admissible if $\lambda=\lambda(L)$ for some Dirichlet surface L. If $\lambda(L)=0$, L is closed. If $L\neq0$ and if $L=L_1+L_2$ with L_1, L_2 closed implies $L_1=kL$ for some k with $0\leqq k\leqq1$, L is called basic closed. A generalized surface L is said to be situated in a subset A of R^m if $(L,f)=0$ for all x in a closed subset of A and all J. Finally, L is basic (λ) if L is an extreme point of the set of all functionals with g-boundary λ.

With these definitions, the authors prove the following four theorems. (I) Let λ_0 be admissible. Then a necessary and sufficient condition that L_0 have g-boundary $\lambda(L_0)=\lambda_0$ is that L_0 be the w^* limit of a sequence of linear combinations $\sum k_jL_j$ with positive coefficients of Dirichlet surfaces L_j with $\lambda(L_j)=\gamma_j\lambda_0$, where each γ_j is a positive integer and $\sum k_j\gamma_j=1$. If the dimension m of the x-space R^m is 3, then the γ_j may be taken to be unity. (II) Suppose $m=3$. Then if $\lambda\neq0$ is admissible and L_0 is basic (λ), then L_0 is the w^* limit of a sequence of Dirichlet surfaces with generalized boundary λ. (III) Suppose $m=3$. Then if L_0 is basic closed, kL_0 is the w^* limit of a sequence of closed Dirichlet surfaces for some $k>0$. (IV) Let f_0 be in E^m and A a compact, convex subset of R^m. Then a necessary and sufficient condition that an exact integrand ϕ and an $\epsilon>0$ exist such that $f_0(x,J)\geqq\phi(x,J)+\epsilon|J|$ for all x in A and all J is that either of the following holds: (a) $(L,f_0)>0$ for every basic closed L situated in A; (b) there exists an open set Ω containing A and an $\epsilon_1>0$ such that $(L,f)\geqq\epsilon_1\times$ area $\times$ (L) for every closed polyhedron L situated in Ω.

These theorems are applied to prove several theorems on the situation of closed non-negative linear functionals L, too complicated to reproduce here, as well as the theorems cited in the opening paragraph of this review. As to the first, the known result was as follows: Let $A\subset R^3$ be compact and convex. Consider the family of Dirichlet surfaces L situated in A whose representations reduce on the perimeter of the unit square to the representation of a given simple closed Frechet curve C. Let $f_0\in E^3$. Suppose that f_0 is positive semiregular, i.e. $f_0(x,J)$ is convex in J for fixed x, and that f_0 is positive definite, i.e. $f_0(x,J)>0$. Then there is a surface L in the given family which minimizes (L,f_0). The generalization of the paper under review is as follows. The requirement that f_0 be positive definite may be relaxed, provided one of the following holds: (i) $(L,f_0)>0$ for every basic closed L situated in A; (ii) there is an open set Ω containing A and an $\epsilon_1>0$ such that $(L,f_0)\geqq\epsilon_1\times$ area $\times$ (L) for every closed polyhedron L situated in Ω; or (iii) $f_0(x,J)\geqq0$ for x in A, $f_0(x,J)+f_0(x,-J)>0$ for x in A and $J\neq0$, and $f_0(x,J)>0$ except in a compact subset X of A satisfying the two following conditions: (a) $R^3-U(\epsilon)$ is connected for the ϵ-neighborhood $U(\epsilon)$ of X, provided ϵ is sufficiently small; (b) given x_1 and x_2 in X there is a plane π separating x_1 and x_2 such that $|\pi\cap X|_2=0$.

The generalization of the Milman theorem in question is as follows: Let $A\subset R^m$ be compact. Then every extreme point of the w^* closed convex span $c(\Gamma)$ of a set Γ of non-negative linear functionals L situated in A is the w^* limit of a sequence in Γ. This was already known [Mil'man, Dokl. Akad. Nauk SSSR (N.S.) **57**, 119–122 (1947); MR **9**, 192] in the bounded case. The generalization of the Kreĭn-Milman theorem [Studia Math. **9**, 133–138 (1940); MR **3**, 90] is as follows. Let $A\subset R^m$ be compact and let f_0 be in E^m with x restricted to A. Then if there is a minimum for (L,f_0) over a w^* closed convex set Γ of non-negative linear functionals L situated in A, this minimum is attained at an extreme point of Γ.

J. M. Danskin.

Bellman, R., Glicksberg, I., and Gross, O. The theory of dynamic programming as applied to a smoothing problem. J. Soc. Indust. Appl. Math. **2**, 82–88 (1954).

The authors consider the problem of minimizing

$$J(x)=\int_0^T[(x-r)+\alpha\max(dx/dt,0)]\,dt$$

over the class of absolutely continuous functions x on $(0, T)$ satisfying $r(t) \leqq x(t) \leqq c$, where r is a given differentiable function with a finite set of maxima and minima. The graph of the solution turns out to be made up of a finite number of closed arcs of the graph of r, joined together by horizontal segments, with possibly a segment joining a point $(0, U)$, with $r(0) \leqq U \leqq c$, to an arc of the graph of r.

J. M. Danskin (Washington, D. C.).

Theory of Probability

Have

★Cramér, Harald. The elements of probability theory and some of its applications. John Wiley & Sons, New York; Almqvist & Wiksell, Stockholm, 1955. 281 pp. $7.00.

This book can be approximately described as a simplified and modernized version of the author's "Mathematical methods of statistics" [Princeton, 1946; MR **8**, 39]. Written in the same clear and lucid style, it is designed for a less mathematically advanced audience and its scope is substantially smaller. Most of the applications are to statistics, and indeed, a sizeable fraction of the book is devoted to statistics. The presentation is distinctly elementary and its level may be gauged from the fact that the central-limit theorem is explained without proof. Interesting exercises complement the textual material. This is a clear and attractive elementary textbook in probability and statistics, and will be enthusiastically welcomed by teachers of elementary courses.

The reviewer wishes to make one very minor criticism. The author discusses seriously but briefly (pages 80–81) the customary criteria for skewness and kurtosis. Although this is standard statistical material widely promulgated, it is easy to verify that these criteria or any based on moments cannot possibly measure skewness or kurtosis. Indeed, it is difficult to see why it is important to measure these at all. We repeat that this is a very trivial criticism of a very meritorious book and is made here not to reflect on the book but with the hope that this "standard" material may cease to be standard.

J. Wolfowitz (Ithaca, N. Y.).

Have Eng.

★Feller, V. [W.] Vvedenie v teoriyu veroyatnostei i eë priloženiya. (Diskretnye raspredeleniya.) [An introduction to probability theory and its applications. (Discrete distributions.)] Izdat. Inostrannoĭ Literatury, Moscow, 1951. 427 pp. 22.10 rubles.

The original appeared in 1950 [Wiley, New York; MR **12**, 424]. The only deviation of the text consists in the omission of all problems and examples which refer to genetics. Otherwise the translators have been satisfied with rephrasing slightly examples using games (like poker) and things with which the average Russian reader is unfamiliar. There are also occasional short explanatory notes by the translators and an introduction by A. N. Kolmogorov.

Have

★Borel, E., Deltheil, R., et Huron, Roger. Probabilités, erreurs. 9ème éd. Librairie Armand Colin, Paris, 1954. 220 pp. 250 francs.

This is the ninth, somewhat modernized, edition of a book which first appeared in 1923. It remains in the classical tradition, however, as can be seen from the chapter headings: Definitions and general principles—Theory of repeated trials. Normal law—Geometrical probabilities—Probabilities of causes. Estimation problems—Errors of observation. Law of Gauss—Method of least squares. Precision of results.

The style is usually clear, and nothing beyond elementary calculus is involved in the mathematics. The applications and examples are too limited to give much insight into the present theoretical and practical scope of probability. There is no indication that the mathematical aspect of probability theory is an integral part of pure mathematics, with its own foundations, rather than a collection of rules and methods for the analysis of certain types of physical phenomena.

J. L. Doob (Urbana, Ill.).

Castoldi, Luigi. Un problema generale di prove ripetute. Rend. Sem. Fac. Sci. Univ. Cagliari **24**, 21–27 (1954).

The schemes of random trials of Bernoulli, Poisson, Lexis, and Coolidge are generalized as follows. The integer l is chosen at random from 1 through k with probability f_l. Then n_l independent trials take place with the probability of success in the ith trial equal to p_{li}. Finally, x is the number of successes observed. The following characteristics of the distribution of x are computed: generating function, factorial-moment generating function, mean, and variance.

L. J. Savage (Chicago, Ill.).

Matschinski, Matthias. Considérations statistiques sur les polygones et les polyèdres. Publ. Inst. Statist. Univ. Paris **3**, 179–201 (1954).

Elementary discussion of relations between, for example, the mean numbers of sides per cell and of sides per vertex when a plane polygonal region is dissected into a large number of polygonal cells. Generalizations to n-dimensional space. Discussion of applications to petrography, etc.

H. P. Mulholland (Birmingham).

Teicher, Henry. An inequality on Poisson probabilities. Ann. Math. Statist. **26**, 147–149 (1955).

The following inequality is proved. If $[\lambda]$ is the greatest integer not exceeding λ, then

$$\sum_{j=0}^{[\lambda]} \lambda^j/j! > \begin{cases} e^{\lambda-1} & \text{for all } \lambda \geqq 0 \\ \frac{1}{2}e^{\lambda} & \text{for all integral } \lambda > 0. \end{cases}$$

R. P. Peterson (Riverside, Calif.).

Zoroa Terol, Procopio. Superposition of random variables and its applications. Trabajos Estadist. **5**, 3–65, 169–216 (1954). (Spanish. English summary)

This is in effect two loosely related papers, corresponding to the break in pagination. The first is on transformations of one real or vector random variable into another, except for a considerable digression on the concept of convergence in distribution as applied to vector random variables. The second is on what the author refers to as the superposition of random variables. If the distribution of ξ is defined in terms of the distribution of η and the distribution of ξ given η, then ξ is said to arise from η by superposition.

The paper is elementary in that it can be read with little training and, considering its length, contains little that would surprise an expert. It is also elementary in the technical sense of sticking to special features of the real and the vectorial (Jacobians and regularity considerations, for example) in contexts where there is also a discipline of abstract measure spaces.

All in all, quite a few remarks and examples seem new to the reviewer. For example ξ_n converges to ξ in probability if and only if (ξ_n, ξ) converges to (ξ, ξ) in distribution.

L. J. Savage (Chicago, Ill.).

Vajda, S. A problem of encounters. Trabajos Estadist. 5, 217–228 (1954). (Spanish summary)

The encounters are between particles of two types, A and B; there are n particles of type A, m of type B and the probability of an encounter in a differential time interval is pdt, for any t and any pair of particles. The author is chiefly interested in the case where an A (B) particle disappears after X (Y) encounters. If both X and Y are infinite, the probability of s encounters is $e^{-\lambda}\lambda^s/s!$, $\lambda=nmpt$; if $X=Y=1$, the probability, $f_s(t)$, of s encounters in time t is the solution of the set of differential recurrence relations

$$df_s(t)/dt+(n-s)(m-s)pf_s(t)=ps^2f_{s-1}(t)$$

with $f_s(0)=\delta_{s0}$ (Kronecker delta). The solution for this, as well as for the mean number of encounters, is given explicitly in terms of binomial coefficients. With X fixed and Y infinite, explicit expressions, too long to quote, are given for the probability and mean of encounters.

J. Riordan (New York, N. Y.).

Ryll-Nardzewski, C. Remarks on the Poisson stochastic process. III. On a property of the homogeneous Poisson process. Studia Math. 14 (1954), 314–318 (1955).

The author considers the realizations of a homogeneous Poisson process and shows that certain random translations produce again a Poisson process. *E. Lukacs.*

Prékopa, András. On compound Poisson distributions. IV. Remarks on the theory of additive processes. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 505–512 (1954). (Hungarian)

Hungarian version of Acta Math. Acad. Sci. Hungar. 3, 317–325 (1953); MR 14, 993.

Takács, Lajos. On secondary processes derived from a Poisson process and their physical applications. With an appendix by Alfréd Rényi. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 473–504 (1954). (Hungarian)

The author considers the following model: The events of a primary process produce a signal depending on time and on a random parameter. It is assumed that the primary process is a (not necessarily homogeneous) Poisson process and that the signal is additive and is determined by a function $f(u, X)$. Here u is the time elapsed since the beginning of the signal and X is the random parameter while $f(u, X)\equiv 0$ if $u<0$. Let $t_1, t_2, \cdots, t_n$ be the moments where events of the Poisson process start a signal and denote by $X_1, X_2, \cdots, X_n$ the corresponding values of the parameter. The total signal at time t is then $Y(t)=\sum f(t-t_k, X_k)$ where the summation is to be extended over all t_k such that $0\leq t_k\leq t$. The author investigates various questions concerning the secondary process $Y(t)$. For instance, he introduces a threshold value for $Y(t)$ and studies the expected time during which the threshold value will be exceeded. A number of examples concerning electron multipliers are discussed. One special case is studied in detail. This case is characterized by the assumptions that the primary process is a time-homogeneous Poisson process and that $f(t, X)=Xe^{-\alpha u}$ ($\alpha>0$, constant). Here X can be interpreted as the random amplitude of a signal which decays exponentially. One of the main results of the paper is the determination of the characteristic function $\phi(t, u)=E[e^{iuY(t)}]$ of the $Y(t)$ process. It is given by $\phi(t, u)=\exp\{-\int_0^t[1-\psi(t-z, u)d\Lambda(z)]\}$. Here $\Lambda(z)$ is the expected number of events in the primary Poisson process occurring during the time interval $(0, z)$ and

$$\psi(t, w)=E\{\exp[iwf(t, X)]\}$$

is the characteristic function of the random variable $f(t, X)$. A second proof of this result is given in an appendix by A. Rényi. Practical applications of some of the results were given by the author in an earlier paper [same Közl. 2, 135–151 (1954); MR 16, 379]. *E. Lukacs.*

Takács, Lajos. On stochastic processes connected with certain physical registration mechanisms. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 571–587 (1954). (Hungarian)

This is a continuation of the paper reviewed above with the following modifications: (i) The assumption that the primary process is a Poisson process is replaced by the assumption that the time intervals between signals are independently and identically distributed random variables; (ii) the restriction $f(u, X)=Xe^{-\alpha u}$ is adopted throughout the paper. The author studies the distribution

$$F(t, x)=P[Y(t)\leq x]$$

and gives sufficient conditions for the existence of the limiting distributions $\lim_{t\to\infty} F(t, x)$ and $\lim_{n\to\infty} F(t_n-0, x)$. The limiting distributions satisfy certain integral equations.

E. Lukacs (Washington, D. C.).

Takács, Lajos. On processes of "happenings" generated by a Poisson process. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 525–541 (1954). (Hungarian)

In an earlier paper [same Közl. 1, 371–386 (1951); MR 13, 956] the author studied processes of "happenings" (secondary processes) generated by a Poisson process. This work is continued and extended in the present paper. A system of simultaneous processes is considered, the system is said to be in the state E_k ($k=0, 1, 2, \cdots, m$) at the time u if "happenings" occur in exactly k processes at this time. Let $C(x)$ be the (exponential) distribution of the time spent in the state E_0 and $D(x)$ the distribution of the time spent outside E_0 and denote by $\varphi(s)$ the Laplace-Stieltjes transform of the convolution $C(x)*D(x)$. The author determines, in terms of $\varphi(s)$, the distribution function of the number of transitions $E_0\to E_1$ during the time interval $(0, t)$ as well as the distribution of the total time spent outside the state E_0 during $(0, t)$. The limits, as $u\to\infty$, of the corresponding distributions, taken over an interval $(u, u+t)$ are called the corresponding stationary distributions and are similarly found. Finally the author determines the expected number of transitions $E_0\to E_1$ for this process during $(0, t)$ and uses it to find $\varphi(s)$. *E. Lukacs.*

Takács, Lajos. "Waiting-time" problems treated by means of Markov processes. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 543–570 (1954). (Hungarian)

This is a largely expository treatment of queuing processes with a single server. It is assumed that the service times are independently and identically distributed while the arrival times follow a Poisson distribution. In one section of the paper the weaker assumption of independently and identically distributed interarrival times is made.

E. Lukacs (Washington, D. C.).

Ghurye, S. G. Random functions satisfying certain linear relations. II. Ann. Math. Statist. 26, 105–111 (1955)

[For part I see same Ann. 25, 543–554 (1954); MR 16, 150.] Let $(x(t): t\in(-\infty, \infty))$ be a real random process, stationary in the wide sense, that is, let $E(x(t)^2)$ be finite and

let the auto-correlation

$$(*)\qquad E(x(t+s)x(s)) \quad t \in (-\infty, +\infty)$$

be independent of s. Suppose, in addition, that (*) is continuous, and let there be k real (weight) functions $\alpha_1(h), \cdots, \alpha_k(h)$, continuous on $[0, +\infty)$, and an integer m, such that the weighted samples

$$(**)\quad y(\alpha; h; n) = x((n+k)h) + \alpha_1(h)x((n+k-1)h) + \cdots + \alpha_k(h)x(nh) \quad (-\infty < n < +\infty)$$

are m-correlated for each positive h, that is, let $y(\alpha; h; i)$ and $y(\alpha; h; j)$ be uncorrelated whenever $|i-j| > m$ and h is positive. Let the number k be minimal in the sense that no such non-correlation obtains for any positive h or integer m using fewer than k weights. Then new weights $\alpha_1, \cdots, \alpha_k$ can be chosen such that the samples (**) are $(k-1)$-correlated. $(k-2)$-correlation is possible only if each sample (**) vanishes with probability 1. The roots of the polynomial

$$(***)\qquad x^k + \alpha_1(h)x^{k-1} + \cdots + \alpha_k(h) = p(x, h)$$

are exponentials: $\exp(\lambda_1 h), \exp(\lambda_2 h), \cdots$. The numbers $\lambda_1, \lambda_2, \cdots$ are the roots of a polynomial of degree k with real coefficients independent of h, and their real parts are not positive. The auto-correlation (*) is a linear combination of these exponentials with polynomial coefficients.

Two similar but weaker results are given: in the first, the moments $E(x(t)^2)$ are finite, $x(t)$ is mean-square continuous, stationarity is dropped, and the correlation condition on (**) is strengthened; in the second, $x(t)$ is merely continuous in probability, and the correlation condition is replaced by an independence condition.

Remark: the condition that $x(t)$ be continuous in probability should be added to the hypotheses of the corollary on page 111. *H. P. McKean, Jr.* (Princeton, N. J.).

Doss, Shafik. Sur le théorème limite central pour des variables aléatoires dans un espace de Banach. Publ. Inst. Statist. Univ. Paris 3, 143–148 (1954).

A version of the central-limit theorem for Banach-space-valued random variables is proved. The random variables are mutually independent, and an analogue of the Lindeberg condition is imposed. A weaker result has been proved by Fortet and Mourier [C. R. Acad. Sci. Paris **238**, 1557–1559 (1954); MR **15**, 805]. *J. L. Doob* (Urbana, Ill.).

Doss, Shafik. Sur la convergence stochastique dans les espaces uniformes. Ann. Sci. Ecole Norm. Sup. (3) **71**, 87–100 (1954).

The author discusses the generalizations of the usual types of convergence of a sequence of numerically-valued random variables (measurable functions) to the case when the random variables have values in a space having a uniform structure. The paper would be clearer if it contained a formal definition of the concept of measurability involved. *J. L. Doob* (Urbana, Ill.).

Dananda, P. H. The central limit theorem for m-dependent variables. Proc. Cambridge Philos. Soc. **51**, 92–95 (1955).

Let $X_1, X_2, \cdots$ be a sequence of m-dependent random variables with zero means and bounded variances. Let $G_i(x) = \Pr(X_i \leqq x)$, $S_n = X_1 + \cdots + X_n$, $s_n' = [\mathcal{E}(S_n^2)]^{1/2}$. If, as $n \to \infty$, $\liminf (s_n'^2/n) > 0$ and

$$n^{-1}\sum_{i=1}^{n}\int_{|x|>\epsilon n^{1/2}} x^2 dG_i \to 0$$

for every $\epsilon > 0$ then S_n/s_n' has the limiting standard normal distribution. This is an extension of theorems of Hoeffding and Robbins [Duke Math. J. **15**, 773–780 (1948); MR **10**, 200], the author [Proc. Cambridge Philos. Soc. **50**, 287–292 (1954); MR **15**, 635] and Marsaglia [Proc. Amer. Math. Soc. **5**, 987–991 (1954); MR **16**, 494]. *W. Hoeffding.*

Mihoc, G. Extension de la loi de Poisson pour les chaines de Markov, multiples et homogènes. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 5–15 (1954). (Romanian. Russian and French summaries)

The author finds the characteristic function of the asymptotic limiting distribution of the number of occurrences of a specified state in n trials of an m-state multiple Markov chain with stationary transition probabilities. Here $n \to \infty$ and the transition probabilities vary in a certain way with n. Koopman [Trans. Amer. Math. Soc. **70**, 277–290 (1951); MR **14**, 1100] treated the case of a simple chain, with $m = 2$. The author [same Bul. **4**, 783–790 (1952); MR **15**, 635] has already treated the case of simple chains, but does not mention the possibility of reducing the present case to the earlier one. *J. L. Doob* (Urbana, Ill.).

Brunk, H. D. On the application of the individual ergodic theorem to discrete stochastic processes. Trans. Amer. Math. Soc. **78**, 482–491 (1955).

Let Ω be a measure space, with measure ν, $\nu(\Omega) = 1$. Let T be a point transformation taking Ω into itself, such that the inverse image of a measurable set is a measurable set. It is shown that the sequence $\{(1/k)\sum_0^{k-1} f(T^s\omega),\ k \geqq 1\}$ is convergent almost everywhere, for each function f which is the characteristic function of a measurable set, if and only if the sequence $\{(1/k)\sum_0^{k-1} \int_\Omega |f(T^s\omega)| d\nu,\ k \geqq 1\}$ is convergent, for each such f. If there is convergence, the first sequence is also convergent almost everywhere for each f such that $f(T^s\cdot)$ is integrable for all s, and the inferior limit of the second sequence is finite for each such f. Closely related results have been proved by Dowker [Duke Math. J. **14**, 1051–1061 (1947); MR **9**, 359] and by Dunford and Miller [Trans. Amer. Math. Soc. **60**, 538–549 (1946); MR **8**, 280]. This theorem is then applied to obtain a corresponding theorem for sequences of random variables, too detailed to state here. *J. L. Doob* (Urbana, Ill.).

Bartlett, M. S. Processus stochastiques ponctuels. Ann. Inst. H. Poincaré **14**, 35–60 (1954).

A point stochastic process is a process whose realization at each time consists of a set (possibly infinite) of isolated points in a space of one or more dimensions. For example, the state of a "cascade" process can be specified by representing the energies of individual particles as points on a line. The paper is mainly expository, treating renewal processes, population growth, contagion, density-fluctuations, etc., chiefly by use of the characteristic functional. The treatment of contagion contains apparently new material, taking account of the spatial positions of the individuals concerned. In some cases a single source of infection leads to a Gaussian density of infected individuals. *T. E. Harris* (Santa Monica, Calif.).

Maruyama, Gisirô. On the transition probability functions of the Markov process. Nat. Sci. Rep. Ochanomizu Univ. **5**, 10–20 (1954).

The author considers a Markov process of diffusion type, defined, following Itô [Proc. Japan Acad. **22**, nos. 1–4, 32–35 (1946); MR **12**, 191] as the solution of a stochastic

integral equation. It is supposed that the problem has been normalized to make the dispersion coefficient identically 1. If the other coefficient satisfies a Lipschitz condition in the space variable, uniformly on the compact time interval under consideration, an explicit expression for the transition probability distribution function is derived. Under further differentiability conditions on this coefficient, it is shown that the transition probability distribution function of the process has continuous second [first] partial derivatives in the initial space [time] variable, and satisfies the backward parabolic diffusion equation. The processes defined directly by stochastic integral equations have thus finally been identified with those defined indirectly by means of their transition probability distribution functions, the latter being defined in turn as solutions of the backward diffusion equation. *J. L. Doob* (Urbana, Ill.).

Kendall, David G. Some analytical properties of continuous stationary Markov transition functions. Trans. Amer. Math. Soc. **78**, 529–540 (1955).

The reviewer [same Trans. **52**, 37–64 (1942); MR **4**, 17] treated Markov chains with countably many states, obtaining derivability properties of the transition probability functions of the processes by means of an analysis of the continuity properties of their sample functions. Kolmogorov [Moskov. Gos. Univ. Uč. Zap. **148**, Mat. **4**, 53–59 (1951); MR **14**, 295] obtained these and somewhat stronger results by a direct analysis of the transition probability functions as solutions of the (Chapman-Kolmogorov) functional equations. The reviewer [Stochastic processes, Wiley, New York, 1953; MR **15**, 445] extended his earlier results and methods to the case of Euclidean state spaces, and the author now extends Kolmogorov's results and methods to abstract state spaces. The author also allows the total space probability to be less than 1. *J. L. Doob.*

Dvoretzky, A., Erdös, P., and Kakutani, S. Multiple points of paths of Brownian motion in the plane. Bull. Res. Council Israel **3**, 364–371 (1954).

P. Lévy [Amer. J. Math. **62**, 487–550 (1940); MR **2**, 107] proved that almost every two-dimensional Brownian motion path has double points. The authors proved [Acta Sci. Math. Szeged **12**, Pars B, 75–81 (1950); MR **11**, 671] that this result is also true for three-dimensional paths, but that almost no path in $n>3$ dimensions has double points. In the present paper it is proved that, for $n=2$, almost every path has k-tuple points, for every k, on every parameter interval. *J. L. Doob* (Urbana, Ill.).

Sirao, Tunekiti. On the uniform continuity of Wiener process. J. Math. Soc. Japan **6**, 332–335 (1954).

Let ϕ be a continuous monotone increasing function on $[0, \infty)$, with $\phi(0)=0$. A function satisfies a weak Lipschitz (or Hölder) condition relative to ϕ if, for some $\epsilon>0$, the condition $|t'-t|=h\leqq\epsilon$ implies that $|f(t')-f(t)|\leqq\phi(h)$. Define

$$\phi(c_1,c_2,h)=\{h[2c_1\log(1/h)+c_2\log\log(1/h)]\}^{1/2}.$$

Then Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937] proved that almost every sample function of the Wiener (Brownian motion) process with variance constant 1, on the parameter interval [0, 1], satisfies the above condition relative to ϕ for $c_2=0$, if $c_1>1$, but that almost none satisfies the condition if $c_1<1$. The author extends this result by proving that, if $c_1=1$, almost every sample function satisfies the condition if $c_2>5$, almost none if $c_2<-1$. *J. L. Doob* (Urbana, Ill.).

Obuhov, A. M. The statistical description of continuous fields. Trudy Geofiz. Inst. no. **24(151)**, 3–42 (1954). (Russian)

This paper contains a systematic exposition of the fundamental questions of the theory of the statistical description of continuous fields, which is a basic tool in theoretical investigations of turbulence. The theory may find wider application with the use of statistical methods of investigations in geophysics. *Author's summary.*

Satô, Hiroshi. On the statistics of a code channel. Rep. Univ. Electro-Commun. **6**, 21–25 (1954). (Japanese. English summary)

The capacity and the symbol probabilities of a noiseless code channel with fixed contraints are derived by the use of the generating function. The asymptotic expression for the number of possible different symbol sequences in a large fixed time and for the average number of occurrences of each symbol are obtained by calculating residues of the generating functions. The distribution of the fluctuation of the number of occurrences of symbols is also discussed. *T. Kitagawa* (Fukuoka).

★Huffman, D. A. Information conservation and sequence transducers. Proceedings of the symposium on information networks, New York, April, 1954, pp. 291–307. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1955.

The author defines a 'sequence source', essentially equivalent to a special Turing machine, and defines a 'flow matrix' for such a source; this 'flow matrix' is essentially Turing's 'Standard description' [see, e.g., Proc. London Math. Soc. (2) **42**, 230–265 (1936)]. He also represents this standard description by an oriented graph which he calls a 'Markoff diagram'. Similarly, equivalent to Turing's machine plus initial strip, the author defines a 'sequence transducer', a 'flow matrix', and a 'flow diagram'.

If the transitions and output symbols are not uniquely determined by the states and initial systems, the sequence transducer is called 'probabilistic', which the author identifies with 'noisy' transducer. In this case the author considers the Markoff chain determined by the matrix of transition probabilities. For sequence sources and transducers, this matrix is merely the incidence matrix of the oriented graph.

The author then applies the standard definitions from information theory to define input information, output information, stored information increment, and lost information. In the case of non-probabilistic transducers there results an identity which the author calls the 'information conservation equation': input information = output information plus stored information increment plus lost information. In the noisy case, the adjustment required to maintain this equality is called the 'noise information'. *S. Gorn* (Aberdeen, Md.).

Kuznecov, P. I., Stratonovič, R. L., and Tihonov, V. I. On the effect of electrical fluctuations on a vacuum-tube generator. Dokl. Akad. Nauk SSSR (N.S.) **97**, 639–642 (1954). (Russian)

The probability distribution is computed for the voltage amplitude across the tank circuit of a vacuum-tube generator fed by a noise source. The noise fluctuations are assumed to be slow compared to the self-oscillations of the vacuum-tube circuit which would exist in the absence of the noise source. The Einstein-Fokker equation is used in the derivation. The result shows three states of the oscil-

lator: the quiescent, the transition state, and the state of oscillation. The limits between any two states are given as functions of the tube and circuit parameters and the characteristics of the noise source. *H. A. Haus.*

✶**Kuznetzov, P. I., Stratonovich, R. L., and Tikhonov, V. I. On the action of electric fluctuations of a tube oscillator.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 7 pp. (mimeographed). $3.00.

Translation of the paper reviewed above.

Mathematical Statistics

✶**Ríos, Sixto. Introducción a los metodos de la estadistica. 2.ª parte. [Introduction to the methods of statistics. Part 2.]** Madrid, 1954. iv+pp. 193–434.

The second part of this book is on a more advanced level than Part 1 [2d ed., Madrid, 1952; MR 14, 568]. The table of contents consists of the following chapter (or lecture) titles: Estimation of parameters. Methods for the formation of estimators. Confidence intervals. Tests of statistical hypotheses. Decision functions. Sequential analysis. Estimation of distributions without parameters. Applications to the measurement of [physical] magnitudes; theory of errors; quality control. Analysis of variance. Design of experiments. Regression problems; multidimensional analysis. Finite populations. Time series. Stochastic processes. Appendix: Operations research. There is also a classified bibliography with over 100 titles (mostly of books), but no index. The author has produced an up-to-date introductory textbook (with exercises) that includes a wide survey of modern statistical methods. It is characteristic of his approach that Kolmogorov's axioms for probability make their first appearance in the last chapter. The book is extremely readable, partly owing to careful explanation (often with diagrams) of the ideas involved, and partly to omission of the harder mathematical proofs. Many results are merely stated as holding "under very general conditions". When conditions are stated they are usually adequate: however, the list on p. 200 of conditions for the Cramér-Rao inequality does not include the existence of integrable majorants for the partial derivatives concerned [cf. H. Cramér, Mathematical methods of statistics, Princeton, 1946, p. 479; MR 8, 39]. *H. P. Mulholland.*

Halperin, Max, Greenhouse, Samuel W., Cornfield, Jerome, and Zalokar, Julia. Tables of percentage points for the studentized maximum absolute deviate in normal samples. J. Amer. Statist. Assoc. 50, 185–195 (1955).

Let x_i, $i=1$ to k, be independent normally distributed variates each with mean μ and variance σ^2. The studentized maximum absolute deviation is defined by

$$d=\max_{i=1,\cdots,k}|x_i-\bar{X}|/s,$$

where ms^2/σ^2 is distributed as χ^2 with m degrees of freedom and independent of x_i. Tables of the upper and lower limit for the upper 5% and 1% points of d are given to 3 significant figures when $k=3$ (1)10(5)20(10)40,6 0 and $m=3$(1) 10(5)20(10)40, 60, 120. Examples illustrate the use of the tables. *L. A. Aroian* (Culver City, Calif.).

Petrov, V. V. On the method of least squares and its extremal properties. Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) 8, no. 4(11), 5–27 (1954). (Romanian)

Translated from Uspehi Mat. Nauk (N.S.) 9, no. 1(59), 41–62 (1954); MR 15, 971.

Akaike, Hirotugu. An approximation to the density function. Ann. Inst. Statist. Math., Tokyo 6, 127–132 (1954).

If in a sample of N from a universe with the density function $f(x)$ the sample values are grouped into class intervals of width 2ϵ, giving observed relative frequencies $f_{N\cdot\epsilon}(x)$ in the intervals, the author defines the best choice of ϵ for a given N as that value for which

$$\int_R E[f_{N\cdot\epsilon}(x)-f(x)]^2w(x)\,dx$$

is minimized, where R is the sample space and $w(x)$ is an "appropriate" weight function. For the six combinations of $f(x)=e^{-x}$ $(x\geqq 0)$, $\frac{1}{2}e^{-|x|}$ $(-\infty<x<\infty)$, $1/a$ $(0\leqq x\leqq a)$ and $w(x)=1$ or $f(x)$, best values of N are given in terms of ϵ. Tables are given for N corresponding to selected values of ϵ in each of these cases; curves constructed from them suggest that at least for these $f(x)$'s, "the usual procedure of taking about from 10 to 20 class-intervals for histograms based on samples of about 500 or more may be considered reasonable". *C. C. Craig* (Ann Arbor, Mich.).

Topp, Chester W., and Leone, Fred C. A family of J-shaped frequency functions. J. Amer. Statist. Assoc. 50, 209–219 (1955).

The authors introduce the following class of cumulative distribution functions: $F(x)=a(2bx-x^2)^r/b^{2r}+(1-a)x/b$ for $0\leqq x\leqq b<\infty$, $F(x)=1$ for $x>b$, $F(x)=0$ for $x<0$, where $0<r<1$ and $0<a\leqq 1$. They compute the moments and give tables for use in choosing a member of the class to fit given empirical data. *L. Weiss* (Charlottesville, Va.).

Vincze, István. Determination of distributions with the aid of mean values. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 513–523 (1954). (Hungarian)

Let $f(x)$ be a frequency function and consider a sequence of subdivisions

$$(1)\qquad a<x_{n1}<x_{n2}<\cdots<x_{nn}<b\quad (n=1,2,\cdots)$$

of the interval (a, b). Suppose that the quotient

$$(2)\qquad \left[\int_{x_{nk}}^{x_{n,k+1}} tf(t)\,dt\right]\left[\int_{x_{nk}}^{x_{n,k+1}} f(t)\,dt\right]^{-1}$$

is known for each subinterval $(x_{nk}, x_{n,k+1})$ of the sequence (1). The author gives conditions which insure that $f(x)$ is determined if the ratio (2) is given for all n and k $(1\leqq k\leqq n)$. The analogous three-dimensional problem is also discussed. *E. Lukacs* (Washington, D. C.).

✶**Uzgören, Nakibe T. The asymptotic development of the distribution of the extreme values of a sample.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 346–353. Academic Press Inc., New York, 1954. $9.00.

Let X be a random variable with a twice differentiable distribution function $P(x)$ such that $P(x)<1$ for all real x, and let $p(x)=P'(x)$, $g(x)=[1-P(x)]/p(x)$. The distribution function $[P(x)]^n$ of the largest of n independent observations of X is studied. Defining the sequence $\{x_n\}$ by

the equations $1-P(x_n)=1/n$, $n=1, 2, \cdots$, the author obtains an asymptotic expansion for $\log[-\log P^n(x)]$ where $x=x_n+ug(x_n)$. He shows that if $\lim_{x\to\infty} g'(x)=0$ and $\lim_{n\to\infty}[ng'(x_n)]=\infty$ then

$$\lim_{n\to\infty} P^n[x_n+ug(x_n)=\exp[-e^{-u}]$$

(Fréchet distribution). These results are then applied to the normal distribution. *Z. W. Birnbaum* (Seattle, Wash.).

Pachares, James. Note on the distribution of a definite quadratic form. Ann. Math. Statist. **26**, 128–131 (1955).

The paper deals with the expansion of the cumulative distribution function $F_n(t)$ of the stochastic variables $Q_n=\frac{1}{2}\sum_1^n a_iX_i^2$, where the X_i are independent $N(0,1)$ variates and where $a_i>0$ for $i=1, 2, \cdots, n$, into an alternating convergent series in such a way that

$$F_n(t)=\frac{t^{n/2}}{|a_1a_2\cdots a_n|^{1/2}}\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\frac{E\{Q_n^{*k}\}}{\Gamma(\frac{1}{2}n+k+1)},$$

where $Q_n^*=\frac{1}{2}\sum_1^n a_i^{-1}X_i^2$. This series converges absolutely and is such that if we stop with any even power of the series we have an upper bound and if we stop with any odd power of the series a lower bound to $F_n(t)$. This result is applied to the distribution of a sum of squares in dependent variables and to that of a sum of independent central chi-square distributions. *T. Kitagawa* (Fukuoka).

Gurland, John. Distribution of definite and of indefinite quadratic forms. Ann. Math. Statist. **26**, 122–127 (1955).

In this continuation of his paper, the author discusses an approximation of the distribution of definite and indefinite quadratic forms in normally distributed variables by means of convergent Laguerre expansions. The problem is reduced to that of the expansion of the cumulative distribution function of $\sum_1^n \lambda_iX_i^2$, where λ's are real numbers and X has the probability density function $f(x)=(2\pi)^{-n/2}\exp(-\frac{1}{2}xx')$. The author establishes the convergent expansions without the restriction imposed by Bhattacharya that all the λ's be positive and that they satisfy the condition $\lambda_k<2n^{-1}\sum_{j=1}^n\lambda_j$ $(k=1, 2, \cdots, n)$, but the present result applies to the cases when the number of positive or the number of negative λ is even. *T. Kitagawa* (Fukuoka).

Korolyuk, V. S. On the discrepancy of empiric distributions for the case of two independent samples. Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 81–96 (1955). (Russian)

The author's summary states: "We obtain the distribution function (d.f.) of the one-sided and two-sided maximal deviation between two empiric d.f.'s from independent samples, the d.f. of the two-sided deviation being obtained under the hypothesis that one sample number is divisible by the other. Besides this, we obtain the d.f. of Kolmogoroff's test of goodness of fit for a finite number of observations in the sample."

The author's d.f.'s are given in terms of finite series. Representations of the last d.f. described above (and related distributions) as an integral and as a determinant were given by A. Wald and J. Wolfowitz [Ann. Math. Statist. **10**, 105–118 (1939)]. *J. Wolfowitz* (Ithaca, N. Y.).

Shimada, Shozo. Power of R-chart. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **3**, 70–74 (1954).

The author is interested in the use of the R chart as an auxiliary to the chart in which only one item per lot is sampled in order to determine whether there has been a change in the means in four successive lots. Let X_i be an observation from a normal population with mean m_i and common variance σ^2, $i=1, 2, 3, 4$. Define the sample range R as the difference between the largest and smallest values of the four observations. The distribution of R is obtained under various assumptions concerning m_1, m_2, m_3, and m_4. The power of this R chart in showing that the values of m_i have changed is computed for various choices of control limits. *L. A. Aroian* (Culver City, Calif.).

Smith, C. D. Tchebycheff inequalities as a basis for statistical tests. Math. Mag. **28**, 185–195 (1955).

Masuyama, Motosaburo. On the error in crop cutting experiment due to the bias on the border of grid. [Application of integral geometry to areal sampling problems. IV.] Sankhyā **14**, 181–186 (1954).

The author applies formulae obtained previously [Sankhyā **12**, 291–302 (1953), §3; Rep. Statist. Appl. Res. Union Jap. Sci. Engrs. **2**, no. 4, 113–119 (1953), §4; MR **15**, 332] in order "to estimate the order of bias in the circular cutting method of estimating crop yield due to over- or underestimation of numbers of plants on the boundary of the circle". *H. P. Mulholland* (Birmingham).

Bhattacharyya, A. Notes on the use of unbiased and biased statistics in the binomial population. Calcutta Statist. Assoc. Bull. **5**, 149–164 (1954).

Let x be the number of successes in n binomial trials, p being the success probability. If $\tau(p)$ is a polynomial in p of degree $\leqq n$, it is shown how to construct a function $t(x)$ such that $E(t)=\tau$; t is a polynomial in x of degree $\leqq n$ and is a minimum-variance estimator, since any function $t(x)$ is a minimum variance estimator of its expectation in binomial estimation. If τ is not of the above form, no minimum variance estimator exists; a number of estimates are given which minimize the bias according to some criterion. *T. E. Harris* (Santa Monica, Calif.).

Patnaik, P. B. A test of significance of a difference between two sample proportions when the proportions are very small. Sankhyā **14**, 187–202 (1954).

The author considers the problem of testing the hypothesis that $m_1=m_2$ when sampling from two Poisson populations. The test is constructed in such a way that the level of significance is made very close to α, instead of being less than or equal to α. The power function of the test is considered and a general method of constructing unbiased tests of a special type for discrete distributions is developed. *L. A. Aroian* (Culver City, Calif.).

Sarhan, A. E. Estimation of the parameters of a skewed distribution by linear systematic statistics. J. Amer. Statist. Assoc. **50**, 196–208 (1955).

Best linear unbiased estimates based on order statistics are given for the mean and standard deviation of a special skewed distribution. Calculations are limited to the case where the sample size $n=2, 3, 4, 5$. It is shown that the midrange and range are highly efficient estimators of the mean and standard deviation for samples of size 5 and smaller. Comparison is made with previous work of this kind for symmetrical distributions. To find the coefficients of the best linear estimator requires the inversion of the variance-covariance matrix of order statistics. The author gives an easy way of performing this inversion and illustrates the method by a numerical example. *B. Epstein.*

Doss, Shafik. Sur une estimation exhaustive pour la moyenne d'une variable aléatoire obéissant à la loi de Laplace dans un espace de Banach. Publ. Inst. Statist. Univ. Paris **3**, 135–142 (1954).

Let $x^{(1)}, \cdots, x^{(n)}$ be mutually independent normally distributed Banach-space valued random variables with a common distribution. If the range space is the line, it is classical that the average is a sufficient statistic for estimating the mean. The corresponding result is proved in the general case. *J. L. Doob* (Urbana, Ill.).

Matusita, Kameo, Suzuki, Yukio, and Hudimoto, Hirosi. On testing statistical hypotheses. Ann. Inst. Statist. Math., Tokyo **6**, 133–141 (1954).

The authors consider five problems involving the normal distribution in which they test a composite hypothesis against a simple or composite alternative. They give bounds for the errors as functions of the distance between two distributions introduced by K. Matusita [same Ann. **3**, 17–35 (1951); MR **13**, 668]. *J. Wolfowitz.*

Siotani, Minoru. An estimate of standard deviation of normal population based on the difference between means of two groups divided by sample mean. Ann. Inst. Statist. Math., Tokyo **6**, 153–160 (1954).

Let $x_1'', \cdots, x_\nu''$ be the order statistics not greater than the mean of a sample of size n from a normal population $(0, \sigma)$, and let $\bar{\bar{x}}(\nu)$ and $\bar{\bar{\bar{x}}}(n-\nu)$ be the means of these order statistics and the remaining ones, respectively. The author derives the distribution of $U_n(\nu)=\bar{\bar{\bar{x}}}(n-\nu)-\bar{\bar{x}}(\nu)$ and compares the estimate $U_n(\nu)/k_{n,\nu}$, where $k_{n,\nu}$ satisfies

$$E[U_n(\nu)\,|\,\nu]=k_{n,\nu}\sigma,$$

with other unbiased estimates of σ based on order statistics. For sample sizes between 12 and 20, e.g., the best estimate based on group ranges [cf. Grubbs and Weaver, J. Amer. Statist. Assoc. **42**, 224–241 (1947)] has a variance which is about 10% larger than that of $U_n(\nu)/k_{n,\nu}$.
D. M. Sandelius (Göteborg).

Cochran, William G. Some methods for strengthening the common χ^2 tests. Biometrics **10**, 417–451 (1954).

A number of methods are presented for strengthening or supplementing the most common uses of the ordinary χ^2 test by (1) using small expectations in computing χ^2, (2) using a single degree of freedom, or a group of degrees of freedom, from the total χ^2, and (3) using alternative tests. The alternative or supplementary tests which are discussed are sensitive to particular kinds of alternatives. Some of them are new, for instance a test of the difference between two proportions in a group of independent 2×2 tables which is developed in an appendix. *W. Hoeffding.*

Pillai, K. C. S. Some new test criteria in multivariate analysis. Ann. Math. Statist. **26**, 117–121 (1955).

In multivariate analysis three problems have especially been studied, viz., (i) testing the hypothesis of equality of the dispersion matrices of two p-variate normal populations, (ii) testing the equality of the p-dimensional mean vectors for l p-variate normal populations (i.e. multivariate analysis of variance), (iii) testing the hypothesis of independence between a p-set and a q-set of variates in a $(p+q)$-variate normal population.

All tests proposed so far for these hypotheses have been shown to depend, when the hypothesis tested is true, only on the characteristic roots of matrices based on sample observations. The choice of any specific function of the roots, as a basis for test criteria, has generally been made heuristically, or purely for convenience in that the sampling distribution of the criterion, under the null hypothesis, was feasible to determine and make available for use.

Here the author gives the approximate distribution of three new criteria, which are harmonic means of the roots, or of simple functions of the roots. The derivation and the determination of the parameter values for which the approximation "is satisfactory for practical use" are based on results of an unpublished paper [K. C. S. Pillai, "On some distribution problems in multivariate analysis", Mimeograph series No. 88, Institute of Statistics, Univ. of North Carolina, 1954]. *D. G. Chapman* (Oxford).

Stoker, D. J. An upper bound for the deviation between the distribution of Wilcoxon's test statistic for the two-sample problem and its limiting normal distribution for finite samples. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 599–606, 607–614 (1954).

Wilcoxon's two-sample statistic U is asymptotically normally distributed under general conditions [see E. L. Lehmann, Ann. Math. Statist. **22**, 165–179 (1951); MR **12**, 726]. The author obtains bounds for the difference between the exact and the asymptotic distribution functions using theorems of A. C. Berry [Trans. Amer. Math. Soc. **49**, 122–136 (1941); MR **2**, 228] and C. G. Esseen [Acta Math. **77**, 1–125 (1945); MR **7**, 312]. *W. Hoeffding.*

Epstein, Benjamin, and Sobel, Milton. Sequential life tests in the exponential case. Ann. Math. Statist. **26**, 82–93 (1955).

This paper describes sequential life test procedures for the special case in which the underlying distribution of the length of life is given by the density

$$f(x, \theta)=\theta^{-1}e^{-x/\theta} \quad (x>0),$$

where $\theta>0$ is the mean life. The primary aim is to test the simple hypothesis $H_0: \theta=\theta_0$ against the simple alternative $H_1: \theta=\theta_1$, where $\theta_1<\theta_0$, with type I and type II errors equal to preassigned values α and β respectively. The test is carried out by drawing n items at random from the population and placing them all on a life test. Both the replacement case in which failed items are immediately replaced by new items and the nonreplacement case are considered. The test can be terminated either at failure times with rejection of H_0, or at any time between failures with acceptance of H_0. Since abnormally long intervals between failures furnish information in favor of H_0 and abnormally short intervals between failure furnish information in favor of H_1, these features are not only reasonable but actually desirable. The authors obtain the likelihood ratio tests and give approximate formulas for the operating characteristic curve, the expected number of failures $E_\theta(r)$, and for the expected waiting time $E_\theta(t)$ before a decision is reached. In the replacement case where the number of items on test throughout the experiment is the same, namely n, it is shown that $E_\theta(t)=(\theta/n)E_\theta(r)$. A table giving approximate values of $E_\theta(r)$ for certain choices of θ/θ_1, α, and β is given for the replacement case. Well chosen examples illustrate the theory. *L. A. Aroian* (Culver City, Calif.).

Dwass, Meyer. A note on simultaneous confidence intervals. Ann. Math. Statist. **26**, 146–147 (1955).

It is shown that to the set of simultaneous confidence intervals given in the analysis of variance by Scheffé [A. Biometrika **40**, 87–104 (1953); MR **15**, 239] and more generally by Roy and Bose [Ann. Math. Statist. **24**, 513-

536 (1953); MR 15, 726], one may append a confidence interval for D where D^2/σ^2 is the so-called "distance" upon which the power of the analysis-of-variance test depends.
H. Teicher (Lafayette, Ind.).

Hajós, György, and Rényi, Alfréd. Some fundamental connections between elementary proofs in the theory of ordered samples. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 467–472 (1954). (Hungarian)

Hungarian version of Acta Math. Acad. Sci. Hungar. 5, 1-6 (1954); MR 15, 972.

Abdel-Aty, S. H. Ordered variables in discontinuous distributions. Statistica, den Haag 8, 61–82 (1954). (Dutch summary)

The author gives formulae for the distribution of individual order statistics, for the joint distribution of two order statistics, and the range in the case where observations come from discrete distributions. The main technical problem involves the treatment of ties. Some detailed calculations are given for the cases where the discrete distribution is binomial or Poisson. Approximations are worked out for the moments of order statistics drawn from these two distributions.
B. Epstein (Detroit, Mich.).

Nordbotten, Svein. On the determination of an optimal sample size. Skand. Aktuarietidskr. 37, 60–64 (1954).

This paper discusses the problem of determination of optimum values of the sample size n and the number of man-hours t spent per sampling unit in processing information when the sampling procedure and total cost are given. In section II a general model is described in which the aim of the survey is to estimate the mean measurement of a characteristic of a finite population when two components of error are present, one due to the sampling error and the other due to the errors in individual measurements. In section III the special case of simple random sampling procedure is discussed and optimum values of t and n are obtained. This section lacks rigor, has several misprints and it is not clear under what assumptions the final result is obtained.
Om P. Aggarwal (Seattle, Wash.).

Neyman, Jerzy. Sur une famille de tests asymptotiques des hypothèses statistiques composées. Trabajos Estadist. 5, 161–168 (1954). (Spanish summary)

Let $(X_1, X_2, \cdots)$ be a sequence of chance variables, and let H denote the hypothesis that the variables are independently and identically distributed, each with probability density function $p(x, \theta)$ of known form but with the value of the parameter θ unknown. To test H with asymptotic probability level α, it is desired to construct a sequence of regions $(A_1, A_2, \cdots)$, A_n being a region in the space of $(X_1, \cdots, X_n)$, such that

$$\lim_{n\to\infty} \int_{A_n} \cdots \int \prod_1^n p(x_i, \theta)\, dx_i = \alpha \quad \text{identically in } \theta.$$

Assuming certain regularity conditions on $p(x, \theta)$, the author constructs such a sequence of regions. The main tool is the central limit theorem.
L. Weiss (Charlottesville, Va.).

Gurland, John. On regularity conditions for maximum likelihood estimators. Skand. Aktuarietidskr. 37, 71–76 (1954).

Let $(x_1, x_2, \cdots, x_n)$ denote a sample of independent observations from a population with probability density $p(x/\theta)$, where θ is an unknown parameter. Further let $L = \prod_{a=1}^n p(x_a/\theta)$. Cramér [Mathematical methods of statistics, Princeton, 1946; MR 8, 39] was one of the first authors to give a rigorous proof of the fact that a root of the equation $\partial \log L/\partial\theta = 0$ is an asymptotically normal and asymptotically efficient estimator of θ. In this paper the author weakens slightly the restrictions Cramér imposed; in particular he shows that it is not necessary to assume the existence of, or conditions on $\partial^3 \log p(x/\theta)/\partial\theta^3$, to prove this result.
D. G. Chapman (Oxford).

Birnbaum, Allan. Characterizations of complete classes of tests of some multiparametric hypotheses, with applications to likelihood ratio tests. Ann. Math. Statist. 26, 21–36 (1955).

For the problem of testing a simple hypothesis concerning parameters of a Darmois-Koopman type density function, explicit characterizations of a minimal essentially complete class of tests, the minimal complete class and the closure of the class of Bayes' solutions are given under certain assumptions. Applications are made to discrete distributions of this type and to some problems of testing composite hypotheses. The likelihood-ratio tests of these hypotheses are characterized and shown to be admissible under certain conditions. (From the author's summary.)
H. Teicher.

Darling, D. A. The Cramér-Smirnov test in the parametric case. Ann. Math. Statist. 26, 1–20 (1955).

Let $X_1, X_2, \cdots, X_n$ be n independent identically distributed random variables having an absolutely continuous distribution function $G(x)$. Let I be a nondegenerate interval on the real axis and suppose, for each θ contained in the interior of I, that $F(x;\theta)$ is a distribution function. The problem is to test the hypothesis H, i.e. H: $G(x) = F(x;\theta)$ for some unspecified $\theta \varepsilon I$. Let $F_n(x)$ be the empirical distribution function based on a random sample of size n. The following test function C_n^2 is considered where

$$C_n^2 = n\int_{-\infty}^{\infty} [F(x) - F(x;\hat\theta_n)]^2 dF(x;\hat\theta_n),$$

and where $\hat\theta_n$ denotes a suitable estimator for θ. H will be rejected for C_n^2 suitably large. When the hypothesis completely specifies the distribution function, i.e. a specific value for θ is given, the C_n^2 test function is the ω^2 test function developed by Cramér, Smirnov, and von Mises. This extension of the ω^2-test to include the parametric case is unlike the ω^2-test in that in general this test is not distribution free when H is true.

When H is true the characteristic function of the limiting distribution of the C_n^2 test function is investigated for several classes of estimators for θ. In particular, subject to given regularity conditions on $F(x;\theta)$ and subject to conditions on $\hat\theta_n$ which include that $nE\{(\hat\theta_n - \theta)^4\} \to 0$ as $n\to\infty$ (where $E\{\cdots\}$ is the expectation operator), the asymptotic characteristic function of the test is given in terms of a Fredholm determinant of a specific integral equation when the method of estimation for θ is by the method of maximum likelihood. The results are obtained by reducing the problem to studying a Gaussian stochastic process in a manner similar to that employed in the paper by T. W. Anderson and D. A. Darling [same Ann. 23, 193–212 (1952); MR 14, 298].
M. Muller (Ithaca, N. Y.).

Terpstra, T. J. A non-parametric test for the problem of k samples. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indag. Math. 16, 505–512 (1954).

In an earlier paper [Math. Centrum Amsterdam, Rapport S92(VP2) (1952); MR 14, 888] the author showed that a k-sample statistic T^2 has a limiting χ^2 distribution when the parent distributions are identical. In the present paper (where T^2 is denoted by Q) a simpler proof of this result is presented. The method is similar to that used by F. Andrews [Ann. Math. Statist. 25, 724–736 (1954); MR 16, 384] in deriving an analogous result for the related statistic H. (The author states only that Q is approximately distributed as χ^2 since he misses a lemma which would imply that this is the limiting distribution. This gap can be bridged as in Andrews' proof.) Tables of the exact distribution of Q are given for some small sample sizes and a comparison with the limiting distribution is made. *W. Hoeffding.*

Kudō, Hirokichi. Dependent experiments and sufficient statistics. Nat. Sci. Rep. Ochanomizu Univ. 4, 151–163 (1954).

If a statistic is adequate for testing, i.e., for all 2-act decision problems, is it necessarily a sufficient statistic? The affirmative is demonstrated in case the underlying class of distributions is finite. This is an advance beyond a demonstration by G. Elfving [Ann. Accad. Sci. Fenn. Ser. A.I. no. 135 (1952); MR 14, 998], which assumed also that each statistic is confined to a finite set of values. Convex sets, and especially "integrals of convex set-valued functions" are employed. *L. J. Savage* (Chicago, Ill.).

Bahadur, R. R., and Lehmann, E. L. Two comments on "Sufficiency and statistical decision functions." Ann. Math. Statist. 26, 139–142 (1955).

An outgrowth of an earlier paper by Bahadur [same Ann. 25, 423–462 (1954); MR 16, 154]. A statistic, in this field, means a function T from a measurable space to an abstract set. The class of measurable sets of the form $T^{-1}(B)$ is a subfield (i.e., sub σ-algebra) of the measurable space, called the subfield induced by T.

It is shown, first, that there exists a subfield necessary and sufficient for a class of measures, that is not induced, even approximately, by any statistic. Second, a subfield is induced by a statistic if and only if no smaller subfield induces the same equivalence classes of points of the space with respect to membership. Finally, some light is cast on the unsolved problem whether the existence of a necessary and sufficient subfield implies the existence of a necessary and sufficient statistic. *L. J. Savage* (Chicago, Ill.).

LeCam, L. An extension of Wald's theory of statistical decision functions. Ann. Math. Statist. 26, 69–81 (1955).

The author's introduction states: "The material of the present paper was developed . . . primarily to meet pedagogical needs. It is similar to the contents of Chapters 2 and 3 of Wald's book [Statistical decision functions, Wiley, New York, 1950; MR 12, 193]. The results are an extension of Wald's theory in the sense that some requirements of boundedness or even finiteness of the loss function are removed. Moreover, Wald's requirements of equicontinuity are replaced by a requirement of lower semicontinuity of the loss function. . . . The methods of proof differ very little from the methods used by Wald, though it has been necessary to use slightly more general topological methods, for instance, to prove compactness instead of sequential compactness."

M. N. Ghosh [Sankhyā 12, 8–26 (1952); MR 14, 1104] earlier had removed the restriction of boundedness of the weight function; the present author states that his own assumptions are weaker than Ghosh's. Some of the results of the present paper are related to results of J. Kiefer [Ann. Math. Statist. 24, 70–75 (1953); MR 14, 998]. *J. Wolfowitz* (Ithaca, N. Y.).

Hannan, James F., and Robbins, Herbert. Asymptotic solutions of the compound decision problem for two completely specified distributions. Ann. Math. Statist. 26, 37–51 (1955).

"A compound decision problem consists of the simultaneous consideration of n decision problems having identical formal structure. Decision functions are allowed to depend on the data from all n components. The risk is taken to be the average of the resulting risks in the component problems. The present paper considers the class of problems where the components involve decision between any two completely specified distributions, with the risk taken to be the weighted probability of wrong decision. For all sufficiently large n, decision functions are found whose risks are uniformly close to the envelope risk function of "invariant" decision functions." (From the author's summary.) *J. Wolfowitz* (Ithaca, N. Y.).

Matusita, Kameo. Decision rule by probability ratio. Ann. Inst. Statist. Math., Tokyo 6, 143–151 (1954).

The author proves very special cases of results on Bayes and minimax decision rules originally due to Wald. Although the author seems to assert that his bounds for the probabilities of the two types of error in a truncated Wald sequential probability ratio test are better than the bounds given by Wald, no evidence to support this assertion is given. *L. Weiss* (Charlottesville, Va.).

Bellman, Richard. Decision making in the face of uncertainty. I. Naval Res. Logist. Quart. 1 (1954), 230–232 (1955).

It is shown in this paper that for a number of simple multi-stage decision processes the intuitive concept of maximizing expected gain over expected cost at each stage is approximately correct. (Author's summary.) *L. J. Savage* (Chicago, Ill.).

Chakravarti, I. M. On the problem of planning a multistage survey for multiple correlated characters. Sankhyā 14, 211–216 (1954).

From a p-variate population a sample is drawn consisting of N_1 first-stage units and N_2 second-stage units in each first-stage unit. Let $\mathbf{x}_{ij}$ denote the vector of p measurements on the jth second-stage unit in the ith first-stage unit. The author considers the model $\mathbf{x}_{ij}=\boldsymbol{\mu}+\boldsymbol{\beta}_i+\boldsymbol{\epsilon}_{ij}$, where $E(\boldsymbol{\beta}_i)=E(\boldsymbol{\epsilon}_{ij})=E(\boldsymbol{\beta}_i'\boldsymbol{\epsilon}_{ij})=0$, $E(\boldsymbol{\beta}_i'\boldsymbol{\beta}_i)=\mathbf{\Lambda}_1$, $E(\boldsymbol{\epsilon}_{ij}'\boldsymbol{\epsilon}_{ij})=\mathbf{\Lambda}_2$, $i=1,\cdots,N_1$, $j=1,\cdots,N_2$. Using the covariance matrix $\mathbf{V}$ of the sample means corresponding to the elements of $\boldsymbol{\mu}$, the author presents three different procedures of determining N_1 and N_2 for the estimation of $\boldsymbol{\mu}$. 1) Minimize $|\mathbf{V}|$ subject to cost restrictions. 2) For each N_1 and N_2 choose that linear compound of the sample means which has maximum variance. Then minimize this variance with respect to N_1 and N_2, under cost restrictions. 3) Same as 1 except that all elements of $\mathbf{V}$ outside the diagonal should be substituted by zeros. The elements of the matrices $\mathbf{\Lambda}_1$ and

A_2 are assumed to be estimated by means of a multivariate variance component analysis performed on data from a preliminary two-stage sample. A numerical illustration is given. *D. M. Sandelius* (Göteborg).

Muhsam, H. V. A probability approach to ties in rank correlation. Bull. Res. Council Israel **3**, 321–327 (1954).

Patankar, V. N. The goodness of fit of frequency distributions obtained from stochastic processes. Biometrika **41**, 450–462 (1954).

Let $X_1, X_2, \cdots, X_n$ be a sequence of dependent or independent random variables sharing a common and known marginal distribution and let the corresponding observed values be $x_1, x_2, \cdots, x_n$; let the x's be ordered by size and grouped so that n_s is the number of observations falling into the sth of r groups. Let $m_s \equiv E(n_s)$; in certain circumstances the $y_s \equiv n_s - m_s$ $(s=1, 2, \cdots, r)$ will have a multivariate normal distribution and if $S \equiv y_s^2/m_s$, then in the case of independent X's we can approximately test S as a χ^2 with $r-1$ degrees of freedom. In general, however, $A \equiv E(S)$ and $2B \equiv \mathrm{var}\ (S)$ will depend on the m's and on quantities like $P^s{}_{ij}$ (the probability that X_t will be in the ith and X_{t+s} in the jth group). The author's proposal is to calculate AS/B and test this as if it were a χ^2 having A^2/B degrees of freedom. Formulae facilitating the calculation of A and B are given for three important cases: (i) the simplest Markovian emigration-immigration process (here called the Poisson-Markov process); the author analyses a sequence of observations made by Westgren on the behaviour of particles in a colloidal solution; (ii) the normal Markov process; (iii) a two-dimensional normal Markov process in which the autocorrelation ρ_{st} is of the form $\rho_1^{|s|}\rho_2^{|t|}$. He applies his formulae for the case (iii) to the analysis of a wheat uniformity trial. *D. G. Kendall* (Oxford).

Whittle, P. On stationary processes in the plane Biometrika **41**, 434–449 (1954).

The author here continues his work on stationary time-series by considering the analysis of stationary famil es of random variables indexed by two "time" parameters and illustrates his arguments by discussing uniformity data for wheat and oranges. Thus in his second example a thousand orange trees were arranged in a 20×50 rectangular lattice and the yield of each tree was considered to be a random variable, the whole family of random variables being indexed by the "time" parameters (s, t) locating the position of a tree in the lattice. The author examines the data in relation to stochastic partial difference equations such as

$$\xi_{st} = \alpha\xi_{s+1,\,t} + \beta\xi_{s-1,\,t} + \gamma\xi_{s,\,t+1} + \delta\xi_{s,\,t-1} + \epsilon_{st},$$

he fits several such schemes and examines the goodness-of-fit. The first eight pages of the paper contain a highly condensed account of a general theory of 2-parameter stationary processes, including discussions of "unilateralisation" and sampling theory. The reviewer found this difficult to follow but has no doubt of the importance of the new ideas being sketched here. There are also some very interesting remarks at the end of the paper concerning an analogous treatment of some stochastic partial differential equations. *D. G. Kendall* (Oxford).

Joshi, D. D. Les processus stochastiques en démographie. Publ. Inst. Statist. Univ. Paris **3**, 153–177 (1954).

The author surveys previous work on deterministic and stochastic models of population growth from the standpoint of a demographer. He remarks that the problem of estimating the parameters of a simple "birth-and-death" process is greatly simplified if the data represent the complete life-histories of a number of individuals instead of the history of the population-size for a fixed interval of time. In concluding sections special attention is given to the problem of accounting separately for the two sexes [cf. L. A. Goodman, Biometrics **9**, 212–225 (1953); MR **14**, 1105], and to the effect of age-dependent fertility and mortality rates. *D. G. Kendall* (Oxford).

Rudra, A. A critical survey of some test methods in time series analysis. Calcutta Statist. Assoc. Bull. **5**, 165–177 (1954).

Expository paper. The criticism is rather formalistic, ignoring such a fact that the methods developed by P. Whittle have proved remarkably successful in applied work [see, e.g., Astrophys. J. **120**, 251–260 (1954)]. *H. Wold* (Uppsala).

Bartlett, M. S. Problèmes de l'analyse spectrale des séries temporelles stationnaires. Publ. Inst. Statist. Univ. Paris **3**, 119–134 (1954).

For Grenander and Rosenblatt's approach towards testing the spectral function of a stationary process [Ann. Math. Statist. **24**, 537–558 (1953); MR **15**, 448] a slight but interesting modification is tentatively discussed and, with numerical illustration, employed for discriminating between the hypotheses of a strictly periodic component vs. an autoregressive process. *H. Wold* (Uppsala).

***Wold, H. Some artificial experiments in factor analysis.** Uppsala Symposium on Psychological Factor Analysis, 17–19 March 1953, pp. 43–64. Ejnar Munksgaard, Copenhagen; Almqvist and Wiksell, Stockholm, 1953. 10 Danish crowns; 7.50 Swedish crowns.

Three sampling experiments were conducted as a test of the Young-Whittle least-squares method of factor analysis [Whittle, Skand. Aktuarietidskr. **35**, 223–239 (1952)]. In each of the first two experiments there were 5 tests, 50 persons, and 2 factors. The factor values were two independent samples, each of size 50, drawn from the positive half of the standard normal distribution (this choice being made to test the method on a skew distribution of factor values). Residuals were normally and independently distributed with zero means, variances being constant (0.16) in the first experiment and proportional to the sum of squares of the factor loadings in the second experiment. The third experiment concerns the simple case of 2 tests, 50 persons and 1 factor. In all three experiments the Young-Whittle method gave satisfactory results. *W. G. Cochran* (Baltimore, Md.).

Delaporte, Pierre. Nouvelle estimation du coefficient de corrélation d'un caractère avec le facteur général ou un facteur de groupe et son écart type, en analyse factorielle. C. R. Acad. Sci. Paris **240**, 1398–1400 (1955).

Nair, K. R. Design and analysis of triangular singly linked blocks. Biometrics **9**, 141–156 (1953).

The designs of the title were described by R. C. Bose and Shimamoto [J. Amer. Statist. Assoc. **47**, 151–184 (1952); MR **14**, 65]. The author describes the analysis of variance and illustrates it by an example. *H. B. Mann.*

Theory of Games, Mathematical Economics

Fleming, W. H. On a class of games over function space and related variational problems. Ann. of Math. (2) **60**, 578–594 (1954).

Consider a game defined as follows. Player I chooses a Borel measurable function x defined on a compact metric space T and with values lying in a compact subset X of k-dimensional Euclidean space. Player II similarly chooses a Borel measurable function y on T with values lying in a compact subset Y of l-dimensional Euclidean space. The payoff is

$$(1) \qquad M(x, y) = \int K(x(t), y(t), t)\, dW.$$

where K is a given continuous function on $X\times Y\times T$ and W is a nonnegative finite nonatomic Borel measure on T. Player I seeks to maximize (1) subject to the side conditions

$$(2_x) \qquad \int \phi_i(x(t), t)\, dW = a_i \quad (i=1, \cdots, m),$$

and player II to minimize (1) subject to side conditions

$$(2_y) \qquad \int \psi_j(y(t), t) dW = b_j \quad (j=1, \cdots, n).$$

The ϕ_i and ψ_j are arbitrary continuous functions.

Let $\mathfrak{X}$ denote the space of x lying in X and satisfying (2_x), and $\mathfrak{Y}$ the space of y lying in Y and satisfying (2_y). This is a game over the function spaces $\mathfrak{X}$ and $\mathfrak{Y}$. As it can be easily seen that pure strategy solutions, i.e. functions $x^0 \in \mathfrak{X}$ and $y^0 \in \mathfrak{Y}$ such that

$$M(x^0, y) \geqq M(x^0, y^0) \geqq M(x, y^0)$$

for all $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$, do not in general exist, the question immediately arises, what is the nature of mixed strategy solutions, if any exist?

Attempts to use general measures over function space as mixed strategies lead to severe difficulties. If a topology is imposed on the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ so as to make them compact and $M(x, y)$ continuous, then every mixed strategy over $\mathfrak{X}$ is, as the author shows, equivalent to a pure strategy. This clearly sharply restricts the class of interesting payoffs. If compactness is relaxed, the question arises, what class of measures should be considered, how should it be described, and in what sense is an existence theorem valid?

In the paper under review the author sidesteps the difficulties just mentioned, by characterizing a mixed strategy as a certain 1-parameter family of functions, as follows. A mixed strategy for player I is a function $x(t, u)$ defined over $T\times(0, 1)$, such that $x_u(t) \in X$ for all t and u, where $x_u(t) = x(t, u)$, and x_u satisfies conditions (2_x) for almost all u. It is understood as follows. Player I chooses a u at random from $(0, 1)$ and plays x_u. Similarly a mixed strategy for player II is a function $y_v(t) = y(t, v)$ on $T\times(0, 1)$ such that $y_v(t) \in Y$ for all t and v, and y_v satisfies (2_y) for almost all v. Player II chooses a v at random and plays y_v. His principal result is that there exist optimal mixed strategies for games of the type (1) under the general conditions described in the first paragraph of this review. That is, there exist mixed strategies $x(t, u)$ and $y(t, u)$ such that

$$\int M(x_u, y)du \geqq \int\int M(x_u, y_v)du dv \geqq \int M(x, y_v)dv$$

for all $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$.

The methods used in the paper are based partly on those of L. C. Young, a mixed strategy in the sense defined above corresponding to a pure strategy in a way analogous to the way in which the generalized curves of Young [C. R. Soc. Sci. Varsovie. Cl. III. **30**, 212–234 (1937)] correspond to ordinary curves of the calculus of variations.

The author also presents a Lagrange-multiplier rule, too complicated to give in a review, by means of which under certain circumstances the game can be reduced to a certain game over the unit square. This reduction, as the author states, is abstract, and further apparatus is needed to provide a description of the solutions $x(t, u)$ and $y(t, u)$.

A number of special cases are of note. If K is concave in x and convex in y, there are pure strategy solutions. If K does not involve y, one easily obtains calculus of variations theorems for the integral

$$(3) \qquad \int_{t_1}^{t_2} K(\dot\gamma(t), t)dW,$$

where we have written $\gamma(t) = x(t)$. In particular, if γ satisfies side conditions $H(\gamma(t_1), \gamma(t_2)) = 0$, $|\gamma(t_1)| \leqq B$, and $\gamma \in \mathfrak{X}$, where H is continuous and possibly vector-valued and $T = (t_1, t_2)$, there is a maximizing $\gamma_0(t)$ for the problem (3). The author gives similar results for parametric problems.

J. M. Danskin (Washington, D. C.).

Isaacs, Rufus. A card game with bluffing. Amer. Math. Monthly **62**, 99–108 (1955).

The author describes and analyzes a zero-sum two-person game with bluffing. Other zero-sum two-person games with bluffing have been analyzed [e.g. J. von Neumann and O. Morgenstern, Theory of games and economic behavior, 2nd ed., Princeton, 1947, pp. 186–219; MR **9**, 50; R. Bellman and D. Blackwell, Red dog, blackjack, and poker, Sci. Amer. **184**, no. 1, 44–47 (1951)]. The game described in this paper is particularly suitable and interesting as an example of the theory for it can involve arbitrarily many moves, it has some recreational appeal, the analysis is not complicated, and the optimal strategies are simple without being trivial.

E. D. Nering.

Bellman, Richard. The theory of dynamic programming. Bull. Amer. Math. Soc. **60**, 503–515 (1954).

This paper presents a "survey of the fundamental concepts, hopes, and aspirations of dynamic programming." It is divided roughly into two parts, mathematical formulation and examples. In the first part, the subheads are: I. A discrete deterministic process; II. Discrete stochastic case; III. Infinite stochastic process; IV. Continuous deterministic process. The subheads of the second part are: I. An allocation problem; II. Stochastic gold mining; III. A problem in the calculus of variations; IV. An eigenvalue problem; V. Games of survival. A bibliography of fifty items is appended.

H. W. Kuhn (Bryn Mawr, Pa.).

***Bernstein, C. "Continuous" programming.** Economic activity analysis, pp. 383–390. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

The author proposes a topological structure for a population of workers based on the idea that if an industry employs a person with certain skills it is likely to employ people with skills close to their skills, closeness being measured in terms of Euclidean distance between points representing numerical scores for their skills.

T. L. Saaty (Washington, D. C.).

Morse, Philip M. Operations research. Comm. Pure Appl. Math. 8, 1–12 (1955).

Modigliani, Franco, and Hohn, Franz E. Production planning over time and the nature of the expectation and planning horizon. Econometrica 23, 46–66 (1955).

Let x_j and s_j be production and known requirements for a commodity in the jth of T time periods, and let X_k and S_k be the corresponding sequences of partial sums. Let h_0 denote the given beginning inventory and $h_k=h_0+X_k-S_k$ the inventory at the end of period k. The problem is so to schedule output that requirements are met, terminal inventory is specified (at zero, say) and costs are minimized. Costs are broken down into fixed costs C_0, production costs $F(x_j)$ and storage costs. The cost function F has a continuous, nonnegative and increasing first derivative. Storage costs are α per period per unit stored. Hence the problem is to choose the x sequence to minimize

$$C_0+\sum_1^T F(x_j)+\alpha\left(\frac{h_0}{2}+\sum_1^{T-1} h_t\right)$$

subject to the constraints $x_j \geqq 0$ and $h_k \geqq 0$. A detailed routine for solution is given which proceeds by solving first the easy unconstrained problem and then making further adjustments. A characteristic of the solution is that it consists of a sequence of unconstrained solutions over successive blocks of periods. *R. Solow* (Cambridge, Mass.).

Mathematical Biology

Komatu, Yûsaku. Distributions of genotypes after a panmixia. J. Math. Soc. Japan 6, 266–282 (1954).

This paper considers the probability distribution of the genotypes occurring in any fixed set of N children derived from a parental population of N females and N males meeting at random, there being Mendelian segregation of m alleles $A_1, A_2, \cdots, A_m$ at a single locus. A generating function is derived giving the joint distribution of the $m(m+1)/2$ numbers C_{ij}, where C_{ij} is the number of children (within the fixed set of N) of genotype A_iA_j. As an application exact formulae are given for the variance of C_{ij} and for the covariance of C_{ij} and C_{kl}. *A. R. G. Owen.*

Komatu, Yûsaku. Mother-child combinations concerning an inherited character after a panmixia. J. Math. Soc. Japan 6, 283–302 (1954).

With the same assumptions as in the preceding paper a generating function is obtained for the joint probability distribution of the $m^2(m+1)^2/4$ numbers $X(ij, kl)$, where $X(ij, kl)$ is the number of mother-child combinations of the type A_iA_j mother with A_kA_l child. An exact formula is derived for the variance of $X(ij, kl)$. *A. R. G. Owen.*

Barricelli, Nils Aall. Esempi numerici di processi di evoluzione. Methodos 6, 45–68 (1954).

Evolution of living organisms is often supposed to be due to the action of natural selection favoring certain combinations of self-reproducing elements (e.g., genes) which arise by random re-arrangement and by random change (mutation). The author claims that these principles are insufficient by themselves, and introduces a third principle of "symbiosis" (roughly, combined action). He illustrates by a process of evolution of artificial "populations", the population $P(r)$ (say) in "generation r" being a series of integers, defined in a simple inductive manner in terms of $P(r-2)$ and $P(r-1)$. On beginning with a random series for $P(0)$, in later generations there are periodic sections, or "species", which can reproduce themselves in subsequent generations, sometimes only by growing "parasitically" on neighbouring species. A kind of symbiosis and non-random mutation is shown by these populations. Experiments with an electronic computer showed the possibility of a limited "evolution" in these cases, though under simple rules it leads either to uniformity or to chaos after a few hundred generations. Possible biological implications of these results are discussed. *C. A. B. Smith* (London).

TOPOLOGY

Trent, Horace M. A note on the enumeration and listing of all possible trees in a connected linear graph. Proc. Nat. Acad. Sci. U. S. A. 40, 1004–1007 (1954).

If N is any elementary node-pair connexion matrix for a connected graph, the number of distinct trees in the graph is the determinant of $N'N$, where N' is the transpose of N. A theorem is also obtained which gives a method of specifying each of these trees. *G. A. Dirac* (Vienna).

Greenwood, R. E., and Gleason, A. M. Combinatorial relations and chromatic graphs. Canad. J. Math. 7, 1–7 (1955).

A graph has N vertices, each pair of distinct vertices being joined by a segment. Each of the segments is given one of the colours $1, 2, \cdots, r$. Let $n(k_1, \cdots, k_r)$ be the least integer such that if $N \geqq n(k_1, \cdots, k_r)$ the graph contains k_i vertices with all interconnecting segments having colour i, for some i $(1 \leqq i \leqq r)$. If $k_1=\cdots=k_r=3$, $n(k_1, \cdots, k_r)=t_r$. The following results are proved:

$$n(3, 3)=6; \quad n(k, m) \leqq n(k-1, m)+n(k, m-1),$$

strict inequality holding if $n(k-1, m)$ and $n(k, m-1)$ are both even; $n(3, 4)=9$; $n(3, 5)=14$; $n(4, 4)=18$; $t_3=17$; $41<t_4 \leqq 66$; $t_r \leqq [(r!)e]+1$. *G. A. Dirac* (Vienna).

Harary, Frank. On the notion of balance of a signed graph. Michigan Math. J. 2 (1953–54), 143–146 (1955).

The edges of a graph are divided into two disjoint classes, called + and − edges. The author establishes several criteria for no circuit to contain an odd number of − edges. *G. A. Dirac* (Vienna).

Kagno, I. N. Corrections to the paper "Linear graphs of degree ≦6 and their groups." Amer. J. Math. 77, 392 (1955).

See same J. 68, 505–520 (1946); MR 8, 46.

de Groot, J. On a compactness criterion of Freudenthal. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 130–131 (1955).

Some remarks on earlier papers of Freudenthal and van Est [same Proc. 54, 295–296, 369–370 (1951); 56, 409–411 (1953); MR 13, 372; 15, 640]. *V. L. Klee, Jr.*

Dinculeanu, Nicolae. Sur les limites généralisées. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 5, 207–214 (1953). (Romanian. Russian and French summaries)

This note gives a slightly different description, by means of ultrafilters, of the consistent scheme of generalized limits

for functions into compact spaces which was defined by Sikorski [Studia Math. 12, 117–124 (1951); MR 13, 216].
M. M. Day (Urbana, Ill.).

Calero, Gonzalo. A topological definition of dimension. Rev. Mat. Hisp.-Amer. (4) 14, 194–199 (1954). (Spanish)

The author bases his definition on the "observation" that if $X\subset H$ and $\overline{H-X}=\bar{H}$ then dim $H>$dim X. If H satisfies this condition only when X is the empty set, H is called an A_0 set. Then, inductively, he defines sets A_n to be sets H which are minimal with respect to $\overline{H-A_{n-1}}=\bar{H}$ for some A_{n-1} and finally defines dimension in terms of an equivalence relation. Since such minimal sets do not exist the definition is not valid. *M. E. Shanks* (West Lafayette, Ind.).

Eggleston, H. G. Covering a three-dimensional set with sets of smaller diameter. J. London Math. Soc. 30, 11–24 (1955).

Borsuk's conjecture that every point set of diameter d in euclidean n-space is a sum of $n+1$ sets each of diameter less than d is proved in this paper for $n=3$. The argument is elementary but quite lengthy. *L. M. Blumenthal.*

Poenaru, Valentin. Sur quelques théorèmes de la topologie plane. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 579–593 (1954). (Romanian. Russian and French summaries)

The author proves two main theorems on finite sets of disjoint Jordan regions in the plane, the first of which is as follows: There does not exist a set of 5 plane Jordan regions, $D_1, \cdots, D_5$ such that a) the regions are disjoint, b) the boundaries of D_i and D_j have at least 3 points in common for each i and j. *M. E. Shanks* (Lafayette, Ind.).

Froda, Alexandru. Ensembles de distances dans l'espace euclidien total. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 5, 29–71 (1954). (Romanian. Russian and French summaries)

A denumerable, ordered set of real numbers is said to define a point, provided that only finitely many of them are different from zero. The set of all points is the total euclidean space $\mathcal{E}$. The euclidean distance between two points is defined as usual. Let E be a set of points of $\mathcal{E}$, such that no two distances between pairs of points are equal and let $D(E)$ be the set of these distances. A set L of positive numbers is called a set of inclusion for E, provided that L is dense in $(0,+\infty)$ and $D(E)\subset L$. For any finite subset of E, the dimension is defined as that of the corresponding vector space and the dimension $r(E)$ of E is defined as lim sup $r(E_m)$, when E_m ranges over all finite subsets of E. A set E is called minimal, provided that each of its subsets contains the least number of points consistent with its dimension. It is shown inductively that if E is minimal and $r(E)=n$ then E contains exactly $n+1$ points; if E is minimal and $r(E)=\aleph_0$, then E contains $\aleph_0$ points. Let $(\mathfrak{L})$ be the class of sets L of real numbers, dense in $(0,+\infty)$ and having the property that any set E of points with unequal distances is minimal provided that $D(E)\subset L$. The main result of the paper is the theorem: $(\mathfrak{L})$ is non-empty. The proof is constructive and uses the axiom of choice. Some further results are: If $L\in(\mathfrak{L})$, $L_1\subset L$ and L_1 is dense in $(0,+\infty)$, then $L_1\in(\mathfrak{L})$; if L is dense in $(0,+\infty)$, then there exists an $L_1\subset L$, such that $L_1\in(\mathfrak{L})$; the set of rationals is not of class $(\mathfrak{L})$. *E. Grosswald* (Philadelphia, Pa.).

Kuratowski, K. Sur une propriété analytique des homéomorphies définies sur des continus plans. Bull. Acad. Polon. Sci. Cl. III. 2, 9–12 (1954).

Kuratowski, K. Fonctions rationnelles qui sont homotopes à des fonctions biunivoques sur certains sous-ensembles du plan. Fund. Math. 41, 107–121 (1954).

Let S_2 be the complex-number plane, compactified at infinity, and let P be the "punctured plane" obtained by deleting from S_2 the points 0 and ∞. For each $A\subset S^2$, let P^A be the space of continuous functions of A into P.

Theorem I. Let $A\subset S_2$ be locally connected. Let $p_0, \cdots, p_n$ be points such that (1) if $i\neq j$, then no continuum in S_2-A contains both p_i and p_j. Let $k_0, \cdots, k_n$ be integers such that (2) $k_i\neq 0$ and (3) $\sum k_i=0$. Let $g\in P^A$ be defined by the equation $g(z)=\prod (z-p_i)^{k_i}$. Suppose that (4) there is a homeomorphism $f\in P^A$, such that f is homotopic to g on A relative to P. Then (5) $\sum|k_n|\leqq 2n$. (Here condition (4) means that the deformation mapping $\phi(z,t)$ ($z\in A$, $0\leqq t\leqq 1$, $\phi(z,0)=f(z)$, $\phi(z,1)=g(z)$) has all its values in P. In the first paper under review, this theorem was proved for the case in which A is a locally connected continuum.)

Theorem II. If $p_0, \cdots, p_n$ are different points of S_2, and $k_0, \cdots, k_n$ are integers satisfying (2), (3) and (5), then there is a set A satisfying (1) and (4). Moreover, A may be chosen so as to be connected, and so as to be the union of n simple closed curves. Or A may be chosen so as to be the union of n sets each of which is homeomorphic to an open annular region bounded by two concentric circles.

Theorem III. Let $A\subset S_2$, and let $f\in P^A$. Suppose that A has only a finite number of components $A_1, \cdots, A_n$; and suppose that for $1\leqq i\leqq n$, $f_i=f|A_i$ is homotopic (on A_i, relative to P) to a rational function. Then f is homotopic (on A, relative to P) to a rational function.
E. E. Moise (Ann Arbor, Mich.).

Pliś, A. Rational functions univalent on sets separating the plane. Bull. Acad. Polon. Sci. Cl. III. 2, 255–256 (1954).

Let $p_0, p_1, \cdots, p_n$ be different points of the complex number plane; and $k_0, k_1, \cdots, k_n$ be integers, such that (1) $\sum k_j=0$, (2) $k_j\neq 0$ for $0\leqq j\leqq n$, and (3) $\sum|k_j|\leqq 2n$. Let $r(z)=\prod (z-p_j)^{k_j}$. Kuratowski in the second paper reviewed above has raised the question whether it follows that there is a curve B such that $r(z)$ is univalent on B and such that B separates the plane between every two different points p_i, p_j. The author gives an example to show that the answer to this question is No. *E. E. Moise.*

Vrublevskaya, I. N. On trajectories and limiting sets of dynamical systems. Dokl. Akad. Nauk SSSR (N.S.) 97, 9–12 (1954). (Russian)

Vrublevskaya, I. N. Some criteria of equivalence of trajectories and semi-trajectories of dynamical systems. Dokl. Akad. Nauk SSSR (N.S.) 97, 197–200 (1954). (Russian)

The author considers dynamical systems $f(p,t)$ defined in a metric space R. Two trajectories L_0 and L_1 are called geometrically equivalent if there is a continuous deformation $F_\lambda(q)$ ($0\leqq\lambda\leqq 1$, $q\in L_0$), of L_0 into L_1 such that (i) for each λ, the set $F_\lambda(L_0)$ is a trajectory L_λ, F_λ is a homeomorphism of L_0 on to L_λ and F_λ^{-1} is uniformly continuous on L_λ, and (ii) $\{F_\lambda\}$, $0\leqq\lambda\leqq 1$, is a uniformly equicontinuous family. Equivalence classes of trajectories are called geometric cells. The trajectories L_0 and L_1 are called kinematically equivalent if also $F_\lambda(f(q,t))=f(F_\lambda(q),t)$ for each $q\in L_0$, $0\leqq\lambda\leqq 1$ and each t. Geometric equivalence is simi-

larly defined for semi-trajectories, ω-limit sets and α-limit sets and kinematic equivalence is defined for semi-trajectories. Geometrically equivalent trajectories are either all closed or all not closed or all singular points.

If R is a complete space then (i) the ω-limit sets of trajectories in the same geometric cell are geometrically equivalent and (ii) geometrically equivalent semi-trajectories are either all wandering or all asymptotic or all stable in the sense of Poisson. If R is the plane and L is a trajectory which approaches spirally its ω-limit set Ω which consists of more than one point then each trajectory geometrically equivalent to L has Ω as its ω-limit point. Let R be the plane and let O_1 and O_2 be singular points and let L_1 and L_2 be trajectories both having O_1 as α-limit set and O_2 as ω-limit set. If the region bounded by $L_1 \cup L_2 \cup O_1 \cup O_2$ contains no singular point and contains no trajectory having either O_1 or O_2 as its common α- and ω-limit set, then L_1 and L_2 are kinematically and hence geometrically equivalent.

In a plane dynamical system defined by a system of differential equations let O be a singular point and let L_1 and L_2 be trajectories both having O as ω-limit set and both having as α-limit set a set Ω consisting of either (i) a cycle or (ii) a finite number of singular points $O_1, \cdots, O_m$, $m \geqq 1$, together with a finite number of trajectories $\Lambda_1, \cdots, \Lambda_n$, $n \geqq 1$, each having one of the points O_i as α-limit set and one of the points O_j as ω-limit set. If the region bounded by $L_1 \cup L_2 \cup O \cup \Omega$ contains no singular points and contains no trajectory having O as its α-limit set then L_1 and L_2 are geometrically equivalent. There is a similar result with O also replaced by a set of the same structure as Ω.

The geometric cells discussed by the author are generalizations of the cells and singular trajectories discussed by Andronov and Pontrjagin [Dokl. Akad. Nauk SSSR (N.S.) **14**, 247–250 (1937)]. No proofs are given in the two papers.

Y. N. Dowker (London).

Tornehave, Hans. On almost periodic movements. Danske Vid. Selsk. Mat.-Fys. Medd. **28**, no. 13, 42 pp. (1954).

Let M be a metric space. A continuous movement in M is defined to be a continuous function on the real line R to M. A diagonal movement in M is defined to be a function $g(a_1t, \cdots, a_nt)$ on R to M where g is a continuous function on R^n to M that has period 2π in each variable, and where $a_1, \cdots, a_n$ are rationally independent real numbers. A generalization of the classical approximation theorem for almost periodic functions is proved, namely, that every continuous movement in M can be uniformly approximated by a diagonal movement in M if M is complete and has the following property: For every compact subset C of M there exists a positive number d such that any two points x and y of C with distance apart less than d can be connected in M by a continuous curve which depends continuously on x and y, and which reduces to x if $x=y$. A similar result is established for families of almost periodic movements. It is shown that almost periodic movements in solenoidal spaces cannot always be so approximated.

The paper also deals with the problem of homotopy between almost periodic movements, offering several theorems and examples. The problem is reduced to the consideration of torus homotopy groups, that is, to the homotopy between maps of tori into the space.

W. H. Gottschalk (Philadelphia, Pa.).

Bagemihl, F. Addendum to "An extension of Sperner's Lemma, with applications to closed-set coverings and fixed points." Fund. Math. **41**, 351 (1955).

See Fund. Math. **40**, 3–12 (1953); MR **15**, 816.

Weier, Josef. Über die wesentlichen Singularitäten einer Abbildungsschar. Math. Ann. **128**, 459–470 (1955).

Let f^α, $\alpha \in I=[0, 1]$, be a continuous family of mappings $P \to P$ where P is a finite polyhedron in R^n. It is assumed that each f^α admits at most a finite number of fixed points. Indices of fixed points are defined by the aid of affine approximations. A point (p, α) in $P \times I$ is called normal if there exists $\epsilon > 0$ and a neighborhood U of p such that for $|t-\alpha| < \epsilon$, f^t has exactly one fixed point in U with non-zero index, that point being p when $t=\alpha$. Let N be the totality of normal points, and consider mappings $t \to (a(t), t) \in N$. Such a mapping of a proper subinterval of I defines an essential singularity if it can not be extended over a larger subinterval. The author proves a number of theorems about the essential singularities. For example, they are disjoint and form a countable set. Each normal point (p, α) is contained in an essential singularity if $0 < \alpha < 1$. If $(a(t), t)$, $t \in J$, defines an essential singularity, then the index of $a(t)$ under f^t is constant over J.

P. A. Smith (New York, N. Y.).

Postnikov, M. M. Definite families of functions and algebras without divisors of zero over the field of real numbers. Uspehi Mat. Nauk (N.S.) **9**, no. 2(60), 67–104 (1954). (Russian)

This is an article of expository character. A definite family of functions consists of n functions f_i of $r+s$ variables $x_1, \cdots, x_r, y_1, \cdots, y_s$, such that (1) the f_i are odd in the x_j and also in the y_k, and (2) the f_i vanish simultaneously only if either all the x_j or all the y_k vanish. The coordinates of the product element $x \cdot y$ in an algebra (over the reals) without zero divisors are such functions (bilinear, with $n=r=s$). After a brief enumeration of the various topological and algebraic approaches to this question, the author describes in §1 the present state of knowledge, in particular, Hopf's condition: all binomial coefficients $C_{n,k}$ with $n-r < k < s$ are even (with the consequence that for $n=r=s$ the number n must be a power of two) [cf. Comment. Math. Helv. **13**, 219–239 (1941); MR **3**, 61; also Stiefel, ibid. **13**, 201–218 (1941); MR **3**, 61], and F. Behrend's upper estimates [Compositio Math. **7**, 1–19 (1939); MR **1**, 36],. §2 develops Hopf's proofs (consideration of induced maps of projective spaces). In the course of this, the author gives an elementary development of the theory of the homology ring of a (polyhedral) manifold and of Hopf's Umkehrungshomomorphisms, coefficients mod 2 (with some of the finer approximation theorems not proved). §3 presents Behrend's proof (by algebraic geometry, valid in all formally-real fields). It contains a description of the necessary basic concepts of algebraic geometry (in particular that of multiple point of a zero-dimensional algebraic manifold and Bézout's theorem).

H. Samelson (Ann Arbor, Mich.).

Kodama, Yukihiro. Mappings of a fully normal space into an absolute neighborhood retract. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5**, 37–47 (1955).

By means of Čech cohomology with local coefficients and slight modification of the reviewer's notions of bridges and bridge mappings, the author proves generalizations of the theorems on continuous extensions and homotopy classifications due to P. Olum [Ann. of Math. (2) **52**, 1–50 (1950); MR **12**, 120], and the reviewer [Trans. Amer. Math. Soc. **64**, 336–358 (1948); MR **10**, 138]. The proofs follow the original ones quite faithfully.

S. T. Hu.

Sitnikov, K. A. Combinatorial topology of nonclosed sets. I. The first duality law; spectral duality. Mat. Sb. N.S. **34**(76), 3–54 (1954). (Russian)

The first chapter of this paper gives a detailed proof of the author's duality law (now called the first duality law) [cf. Dokl. Akad. Nauk SSSR (N.S.) **81**, 359–362 (1951); MR **13**, 860]. The second chapter proves the stronger "spectral duality": The groups $\nabla^p A$ and $\Delta^q B$ can be considered as direct limits of certain spectra (on the star-finite coverings of A and the compact subsets of B as directed sets). The main content of spectral duality is the equivalence of these two spectra in the sense that one can pass from one to the other in a finite number of the following steps: passage to a cofinal subset, the inverse of this, passage to an isomorphic system. The proof makes use of the canonical triangulations and coverings introduced by P. S. Aleksandrov [cf. Mat. Sb. N.S. **33**(75), 241–260 (1953); MR **16**, 503], and of a modification of directed systems called multiplication (one considers any number of copies of the original system, with a suitable order defined in the whole set). It is shown that, if the coefficient group is compact, then the Sitnikov groups $\Delta^q B$ are isomorphic with the usual groups $\Delta_c{}^q B$ with compact carriers. The connection with Aleksandrov's duality law (loc. cit.) is discussed: $\nabla^p A$ and $\Delta^q B$ contain the topologically invariant subgroups $\nabla_0{}^p A$ and $\Delta_0{}^q B$ of "non-linking" cycles [cf. Aleksandrov, loc. cit.]. Under the Sitnikov duality law these two subgroups correspond to each other. Aleksandrov's law says essentially that the factor groups $\nabla^p A - \nabla_0{}^p A$ and $\Delta^q B - \Delta_0{}^q$ are isomorphic.

The third chapter proves stability of the $\nabla^p A$, namely that they are naturally isomorphic to the groups $\nabla^p A_1$ for any general deformation retract A_1 of A ("general" meaning that during the deformation the points of A_1 may move, but have to stay in A_1). Ch. 4: A set A is called a K-set if for any discrete group $\mathfrak{A}$ and its dual $\mathfrak{B}$, the group $\Delta^p(A, \mathfrak{B})$ consists exactly of all characters of $\nabla^p(A, \mathfrak{A})$, with respect to the natural scalar product, and the same with Δ and ∇ interchanged, for all p. A duality domain, in the sense of Aleksandrov, is a family of subsets of the spheres S_n $(n=1, 2, \cdots)$, which is closed under formation of complement (with respect to the containing sphere) and topological image. It is shown that the K-sets form a duality domain, i.e. that the complement B of a K-set $A \subset S_n$ is again a K-set. For K-sets the groups Δ^p and $\Delta_c{}^p$ are isomorphic for any coefficient group. Moreover, the K-sets form an elementary duality domain, i.e. for dual coefficient groups the groups $\Delta_c{}^p(A)$ and $\Delta_c{}^q(B)$, $p+q=n-1$, are character groups of each other, paired by the linking coefficient; the last condition, with $\Delta_c{}^p$ replaced by Δ^p, characterizes K-sets. The set A is called quasi-open, if, for discrete $\mathfrak{A}$ and its dual $\mathfrak{B}$, the groups $\Delta_c{}^p(A, \mathfrak{A})$ and $\Delta_c{}^q(B, \mathfrak{B})$ are dually paired by the linking coefficient; this is a topological property. Interchanging $\mathfrak{A}$ and $\mathfrak{B}$, one gets the quasi-closed sets; their topological invariance is not yet established.

H. Samelson (Ann Arbor, Mich.).

Burdina, V. I. Real characteristic cycles of complex manifolds. Dokl. Akad. Nauk SSSR (N.S.) **96**, 1085–1088 (1954). (Russian)

For a complex manifold the author studies the relation between the real characteristic classes (Pontryagin and Stiefel-Whitney) and the complex characteristic classes (Chern). This is reduced to a study of the natural imbedding of the complex Grassmann manifold $C(k, l)$ [of complex k-spaces in $(k+l)$-space] in the real Grassmann-manifold $H(2k, 2l)$. This in turn involves the intersections of the cycle $C(k, l)$ with the Pontryagin, etc. cycles of $H(2k, 2l)$. Using Pontryagin's description of these cycles [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **35**, 34–37 (1942); MR **4**, 147] the author expresses the cycles as determinants in certain special cycles, and then states a formula giving the intersection of $C(k, l)$ with the individual special cycle. This is done for weak homology (integral, mod torsion; Pontryagin classes) and mod 2 (Stiefel-Whitney classes). The Chern classes reduced mod 2 are shown to be the Stiefel-Whitney classes [for this and other related results, e.g. the formula expressing the Pontryagin classes in the Chern classes, cf. W. T. Wu, Sur les classes caractéristiques des structures fibrées sphériques, Hermann, Paris, 1952; MR **14**, 1112; cf. also the expressions for these classes as symmetric functions, Borel, Comment. Math. Helv. **27**, 165–197 (1953); MR **15**, 244; and Borel and Serre, Amer. J. Math. **75**, 409–448 (1953); MR **15**, 338].

H. Samelson.

Serre, Jean-Pierre. Une propriété topologique des domaines de Runge. Proc. Amer. Math. Soc. **6**, 133–134 (1955).

A domain X in the n-dimensional complex vector space C^n is a Runge domain if X is a domain of holomorphy and every function holomorphic in X can be approximated by polynomials in every compact subdomain of X. The author proves: The nth Betti number of a Runge domain in the C^n is zero and the homology group $H_n(X)$ has only elements of finite order. For the proof the author shows that the nth cohomology group $H^n(X, C)$ is zero.

H. J. Bremermann (Münster).

Serre, Jean-Pierre. Un théorème de dualité. Comment. Math. Helv. **29**, 9–26 (1955).

This is a detailed treatment of the author's celebrated duality theorem, which has numerous applications in the theory of complex manifolds. Let X be a complex manifold of complex dimension n, and V an analytic vector bundle over X, whose fiber is a complex vector space of dimension r. Let $\mathbf{A}^{p,q}(V)$ be the sheaf (faisceau) of germs of C^∞ differential forms of type (p, q) with coefficients in V, and $\Omega^p(V)$ be the sheaf of germs of holomorphic differential forms of degree p with coefficients in V. The set of sections $A^{p,q}(V)$ of the sheaf $\mathbf{A}^{p,q}(V)$ can be given a topology and is a Fréchet space. Let V^* be the dual vector bundle of V. Let $H^q(X, \Omega^p(V))$ and $H_*{}^{n-q}(X, \Omega^{n-p}(V^*))$ be the cohomology groups of X, with closed and compact supports respectively, in which, as usual, the superscripts denote the dimensions and the second entries in the parentheses the coefficient sheaves. The duality theorem states that, if the mappings

$$A^{p,q-1}(V) \xrightarrow{d''} A^{p,q}(V) \xrightarrow{d''} A^{p,q+1}(V)$$

are homomorphisms, the dual topology of the Fréchet space $H^q(X, \Omega^p(V))$ is canonically isomorphic to

$$H_*{}^{n-q}(X, \Omega^{n-p}(V^*)).$$

For $p=0$ the conclusion can be stated by saying that the dual topology of the Fréchet space $H^q(X, \Omega^0(V))$ is canonically isomorphic to $H_*{}^{n-q}(X, \Omega^0(\bar{V}))$, where $\bar{V}$ is an analytic vector bundle constructed from V, with the property that $\bar{\bar{V}} = V$.

This duality theorem has among others the following consequences: 1) If X is a Stein manifold, then

$$H_*{}^q(X, \Omega^p(V)) = 0, \quad q \neq n, \quad \text{and} \quad H_*{}^n(X, \Omega^p(V))$$

is isomorphic to the dual topology of $H^0(X, \Omega^{n-p}(V^*))$. 2) If X is compact, then $H^q(X, \Omega^p(V))$ and $H^{n-q}(X, \Omega^{n-p}(V^*))$ are finite-dimensional and are dual vector spaces. 3) If X is compact and V a complex line bundle defined by a divisor D, then the vector spaces $H^q(X, \Omega^p(D))$ and $H^{n-q}(X, \Omega^{n-p}(-D))$ are dual to each other. The author also gives a simple deduction of the Riemann-Roch Theorem for curves from the duality theorem. The paper gives in its first section a summary of cohomology theory with coefficient sheaves needed for subsequent treatment.

S. Chern (Chicago, Ill.).

Blanchard, André. La cohomologie réelle d'un espace fibré à fibre kählérienne. C. R. Acad. Sci. Paris **239**, 1342–1343 (1954).

Let $E(B, F)$ be a compact connected fiber space, with fiber F and base space B, of which F is connected and pseudo-Kählerian. Suppose that the real cohomology ring $H^*(F)$ and the real cohomology class Q of degree 2 of F defined by the pseudo-Kählerian structure are acted on trivially by the fundamental group $\pi_1(B)$ of B. It is proved that: 1) a necessary and sufficient condition that Q is induced by a cohomology class of E is that the transgression $H^1(F)\to H^2(B)$ is zero; 2) the condition in 1) implies that all the Leray differentials are zero and that the real cohomology groups of E are isomorphic to those of $B\times F$. Proofs are indicated.

S. Chern (Chicago, Ill.).

Nakano, Shigeo. On a certain type of analytic fiber bundles. Proc. Japan Acad. **30**, 542–547 (1954).

The fibre bundles $\mathfrak{B}$ considered have as base a compact complex manifold V, the fibres of affine lines, and the group is the complex affine group. If the coordinate transformations $(z, \zeta_i)\to(z, \zeta_j)$ are given by $\zeta_i = a_{ij}(z)\zeta_j + b_{ij}(z)$, the coefficients (a_{ij}) define a complex line bundle $\mathfrak{A}$ in the sense of K. Kodaira [Proc. Nat. Acad. Sci. U. S. A. **39**, 865–868 (1953); MR **16**, 74], which is said to be subordinate to $\mathfrak{B}$. It is shown that, given $\mathfrak{A}$, the bundles $\mathfrak{B}$ to which it is subordinate are represented by the points of the projective space which corresponds to the rays of the vector space $H^{01}(\mathfrak{A})$. In the case in which V is an algebraic curve, it is shown that this invariant of $\mathfrak{B}$ is the same as that of A. Weil [Fibre spaces in algebraic geometry (mimeographed notes), Univ. of Chicago, 1952].

W. V. D. Hodge.

Boltyanskiĭ, V. The problem of freeing a cross-section from a subbundle. Dokl. Akad. Nauk SSSR (N.S.) **99**, 669–672 (1954). (Russian)

The problem under discussion is the following. Let (P, p, B, F, G) be a fibre-bundle and let (Q, p, B, E, G') be a subbundle in the sense that $Q\subseteq P$, E is invariant under the operations of G and $G'=G/N$, where N is the group of trivial operations on E. Moreover, suppose that B is a simplicial complex and that, for every simplex $T\in B$, the co-ordinate function $\xi_T\colon T\times F\to P^{-1}(T)$ becomes, on restriction to $T\times E$, the co-ordinate function for the subbundle. A cross-section $\sigma: B\to P$ is freed from Q by deformation if there exists a homotopy of cross-sections $\sigma_t: B\to P$ with $\sigma_0=\sigma$, $\sigma_1 B\subseteq P-Q$. The author analyses the obstructions to freeing σ by deformation and obtains a theory which is a relativization of the standard obstruction theory for cross-sections.

The obstruction is formulated precisely as follows. Let σ be a cross-section on B^r mapping B^{r-1} into $P-Q$. Then for every oriented r-simplex T we may define the map ϕ: T, $\dot{T}\to F$, $F-E$, given by $\xi_T(x, \phi x)=\sigma x$. Assuming, to avoid local coefficients, that $(F, F-E)$ is r-simple, we get an obstruction cochain $s^r(\sigma)\in C^r(B; \pi_r(F, F-E))$. The author relates this obstruction to the classical obstructions and also to k-deformations: a k-deformation σ_t between two cross-sections of B^r which are both free on B^{r-1} satisfies $\sigma_t(B^{r-k-1})\subseteq P-Q$. The author proves that $s^r(\sigma)=0$ if and only if σ can be freed on B^r by a 0-deformation and that $s^r(\sigma_0)\sim s^r(\sigma_1)$ if σ_0 is 1-deformable into σ_1 (conversely, given σ_0 and a coboundary, we may 1-deform σ_0 into a cross-section σ_1 such that $s^r(\sigma_0)$ differs from $s^r(\sigma_1)$ by the given coboundary). Results on 2-deformations are also given, under more restrictive conditions; these involve either the Pontryagin square ($r=4$) or the Steenrod square ($r>4$).

If $\pi_i(F, F-E)=0$, $i<r$, then there is no obstruction to freeing an arbitrary cross-section σ on B^{r-1}; the cohomology class of the obstruction to freeing it on B^r depends only on σ and is 0 if and only if σ may be freed on B^r. Any two deformations freeing σ on B_r lead to second obstructions whose difference must have a certain given form, and σ may be freed on B_{r+1} if and only if any second obstruction has this form.

P. J. Hilton (Cambridge, England).

GEOMETRY

***Thébault, Victor. Parmi les belles figures de la géométrie dans l'espace. (Géométrie du tétraèdre.)** Librairie Vuibert, Paris, 1955. xvi+287 pp. 2000 francs.

The founders of the modern geometry of the triangle have also contributed to the geometry of the tetrahedron, particularly J. Neuberg. The plane geometry has caught the fancy of a vast legion of enthusiastic followers, and during the latter part of the last century the subject enjoyed a "mushroom growth" (J. L. Coolidge). The progress of the geometry of the tetrahedron has been relatively slow. The difference is due, in the first place, to intrinsically greater difficulty of the problems in space. But what is more important in the present case, in the geometry of the triangle, as in the elementary theory of numbers, a great deal could be accomplished with quite modest means, and the field was thus wide open to young and old, to the amateur as to the professional.

Among those who in the past quarter of the century have tried to fill the gap between the geometry of the triangle and the geometry of the tetrahedron the author has played a leading role. His numerous contributions have appeared in various journals in many countries. He found for the tetrahedron, analogs for elements of the triangle about which little or nothing was known before.

In the book under review the author has collected a considerable amount of material, largely his own contributions, which have first seen the light of day during the last three decades. Practically none of it appeared in book-form before. In many parts of the book the subject matter is treated synthetically, often elegantly. By and large, however, the method of presentation is computational, with or without coordinates, much more so than is usual with this author in his writings in plane geometry.

Ronald Deaux, editor of Mathesis, the journal which had "la primeur" of many of Thébault's papers, points out in his preface to the book that the author has not attempted to write a treatise on the subject, and no claims of a didactic nature are made for it. But the book will be enjoyed by

those who are interested in the progress of the geometry of the tetrahedron and who appreciate "la belle Géométrie". It is indispensable for those who are working along these lines. The latter class of readers would be helped if the references were more numerous, the table of contents more detailed, if the book had a glossary and a list of names cited, but most of all, if the book had an index. *N. A. Court.*

Mahler, Kurt. A problem in elementary geometry. Math. Gaz. **38**, 241–243 (1954).

The author proves the following theorem: Let T be a triangle whose vertices A, B, C are, respectively, inner points of the sides a, b, c of another triangle t. Then there exists a point D and an arbitrarily small rotation of center D which carries T into the interior of t. The author further remarks that the theorem is false for rectangles, but remains true for simplices of higher dimension. *P. Erdös.*

Cavallaro, Vincenzo G. Triangoli ortogonalmente associati. Giorn. Mat. Battaglini (5) **2(82)**, 423–427 (1954).

✱Mandzyuk, A. I. On triangles inscribed in one conic section and circumscribed about another. Nomografičeskiĭ sbornik [Nomographic collection], pp. 35–38. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

The author derives some theorems of A. Haarbleicher [C. R. Acad. Sci. Paris **206**, 1212–1215 (1938)] concerning Poncelet triangles from results obtained by himself [Trudy Moskov. Zootehn. Inst. im. Molotov. **4**, 133–135 (1936)] and D. D. Morduhaĭ-Boltovskoĭ [Trudy Inst. Mat. Estest. Severo-Kavkazkoĭ Univ. **2**, 29–38 (1926)] which were not accessible to this reviewer. *E. Lukacs.*

Labra, Manuel. Properties of the medians of an inscribed quadrilateral. Rev. Soc. Cubana Ci. Fis. Mat. **3**, 79–88 (1954). (Spanish)

Jarden, Dov. The sums of the dihedral and trihedral angles in a tetrahedron. Riveon Lematematika **8**, 32 (1954). (Hebrew. English summary)

Independent elementary proof of an inequality of J. W. Gaddum [Amer. Math. Monthly **59**, 370–371 (1952); MR 13, 968] reproduced from Dov Juzuk (=Jarden), Technika Umada **26**, 8–9 (1941) (Hebrew). *T. S. Motzkin.*

Devidé, Vladimir. Verallgemeinerung einer Formel von L'Huilier. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 121–127 (1954). (Serbo-Croatian summary)

The formula referred to is: $1/r_0=1/r_1+1/r_2+1/r_3$, where r_0 is the inradius, and r_1, r_2, r_3 the exradii of a triangle. (The author gives the incomplete reference: L'Huilier 1809. The formula was first given, anonymously, in Ann. Math. Pures Appl. **19**, 211–218 (1829).) The author has given a first generalization of this formula [same Glasnik **6**, 145–154 (1951), p. 150; MR 13, 576; 14, 1277]. The restrictions under which that formula was obtained are removed in the present paper. (The author obligingly called to the attention of the reviewer that these restrictions were made explicit in the Serbo-Croatian text. They were not mentioned in the German summary.)

The result arrived at is as follows. If we sum up the reciprocal values $1/r$ of the radii of all possible tangent hyperspheres of an n-dimensional simplex S_n, and affect each value with the sign $+$ or $-$ (that is, $\epsilon=+$ or $-$) according as the corresponding hypersphere touches a larger number of lateral spaces of S_n internally or externally (exclusive, when n is odd, of the reciprocal values of the radii of the hyperspheres which touch the same number of lateral spaces of S_n internally as externally), we obtain

$$\sum \epsilon\cdot\frac{1}{r}=\binom{n}{[\frac{1}{2}n]}\frac{1}{r_0},$$

where r_0 is the radius of the inscribed sphere of the simplex S_n. *N. A. Court* (Norman, Okla.).

Devidé, Vladimir. Einige metrische Relationen über Simplexe. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 115–120 (1954). (Serbo-Croatian summary)

Let $S_{m-2,r}$, $S'_{m-2,r'}$ denote two hyperspheres of euclidean $(m-1)$-space, with radii r, r', respectively, and p the distance between their centers. If $P=(p_1, p_2, \cdots, p_m)\subset S_{m-2,r}$ and $Q=(q_1, q_2, \cdots, q_m)\subset S'_{m-2,r'}$ the author shows that

$$D_m=|s^2{}_{ik}|=(-1)^{m-1}2^{m-1}[(m-1)!]^2(r^2+r'^2-p^2)\Delta_m\cdot\Delta_m',$$

$$G_m=\begin{vmatrix} s^2{}_{ik} & 1 \\ 1 & 0\end{vmatrix}_B=(-1)^m2^{m-1}[(m-1)!]^2\Delta_m\cdot\Delta_m',$$

where $s_{ik}=p_iq_k$ $(i=1, 2, \cdots, m; k=1, 2, \cdots, m)$, Δ_m, Δ_m' denote the volumes of the simplices determined by P, Q, respectively, and $\begin{vmatrix} s^2{}_{ik} & 1 \\ 1 & 0\end{vmatrix}_B$ symbolizes the determinant obtained from D_m by bordering it with a row and column of 1's, with intersecting element 0. Various specializations of these two formulae yield some well-known results.

L. M. Blumenthal (Leiden).

Seidel, Jacob. Angoli fra due sottospazi di uno spazio sferico od ellittico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 625–632 (1954).

The mutual position of two subspaces Γ, Δ of dimensions p, q, respectively ($p\geqq q$) of spherical or elliptic m-space is determined by $q+1$ metric invariants (the angles between Γ and Δ) which are the stationary values of the distance between a point of Γ and a point of Δ. An equation is found whose roots are the cosines of these angles and whose coefficients depend on the distances between the points of bases in Γ and Δ. *L. M. Blumenthal* (Leiden).

Hofmann, Ludwig. Über die Herstellung achsialer Lagen von kollinearen Räumen bei Zugrundelegung einer elliptischen Metrik. Monatsh. Math. **58**, 143–159 (1954).

In a recent paper [Monatsh. Math. **57**, 75–87 (1953); MR 15, 56] H. Brauner discussed, in effect, the general possibility of representing a collineation T of real euclidean space E_3 as a product CB of a biaxial collineation B of E_3 with a congruence transformation C of E_3. This problem is poristic, having no solution for general T and an infinity of solutions if T satisfies a certain condition. In the present paper, the same problem is discussed for a real non-euclidean space R_3 with an elliptic metric, and the general character of the result is very similar. The required necessary and sufficient condition for any factorization $T=CB$ is that, if Φ is the absolute quadric in R_3, the quadrics $T(\Phi)$ and $T^{-1}(\Phi)$ should be congruent; and this condition can be interpreted, with suitable conventions, as the requirement that the (non-euclidean) lengths of four properly chosen semi-axes of $T(\Phi)$ should be equal in pairs. When the condition is satisfied, the problem has in general two ∞^1 systems of solutions; if, in addition, $T(\Phi)$ is a quadric of revolution, it

has ∞^2 solutions; and if $T(\Phi)$ is a Clifford surface, and therefore always congruent to $T^{-1}(\Phi)$, then the problem has two ∞^2 systems of solutions. *J. G. Semple* (London).

Lenz, Hanfried. Über die Einführung einer absoluten Polarität in die projektive und affine Geometrie des Raumes. Math. Ann. **128**, 363–372 (1954).

In a projective space P of arbitrary (possibly infinite) dimension $\geqq 3$, the author introduces the concept of "quasipolarity". This is a 1-1 mapping π of the space into its dual such that if A is on $\pi(B)$, then B is on $\pi(A)$. If P is finite-dimensional, then π is onto, thus a polarity; on the other hand, polarities are possible only if P is finite-dimensional. It is shown that a quasipolarity is representable by a system of linear equations over the coordinate field of P whose matrix is left non-singular, and that the metric concepts of Euclidean and non-Euclidean geometry are naturally introduced by considering points conjugate with respect to the quasipolarity. *A. J. Hoffman* (Washington, D. C.).

Ree, Rimhak. On projective geometry over full matrix rings. Proc. Amer. Math. Soc. **6**, 144–150 (1955).

Die Arbeit befasst sich mit Strukturuntersuchungen über verbandstheoretisch definierte projektive Geometrien über einem Ring bzw. einem vollen Matrizenring im Anschluss an Untersuchungen von R. Baer [Trans. Amer. Math. Soc. **52**, 283–343 (1942); Ann. of Math. (2) **44**, 192–227 (1943); Linear algebra and projective geometry, Academic Press, New York, 1952; MR **4**, 109, 267; **14**, 675]. Zu einem assoziativen Ring R mit Einselement heisst eine additive Gruppe A ein R-Modul, wenn $a(x+y)=ax+ay$, $(a+b)x=ax+bx$, $a(bx)=(ab)x$, $1x=1$ für alle a, $b \in R$, alle x, $y \in A$. Eine Untergruppe B von A heisst R-Untermodul, wenn $ax \in B$ für $a \in R$, alle $x \in B$. Die Menge aller R-Untermoduln von A bildet einen modularen Verband $L(R, A)$, der als eine projektive Geometrie über R bezeichnet wird. Ist R_n der volle Matrizenring über R, so gilt: 1) Zu jedem R_n-Modul M existiert ein R-Modul A, und umgekehrt, so dass $L(R_n, M)$ und $L(R, A)$ isomorph sind. 2) Erfüllt ein System von Ringen $F=(R, S, \cdots)$ bezüglich einer verbandstheoretischen Bedingung den Fundamentalsatz der projektiven Geometrie in verbandstheoretischer Formulierung, so gilt diese Aussage auch für das System $F_n=(R_n, S_n, \cdots)$ für jede natürliche Zahl n. 3). Ist R zusätzlich so beschaffen, dass jeder Verbandsisomorphismus in $L(R, A)$ erzeugt wird durch einen passenden Automorphismus σ von A mit $(ax)^\sigma=a^{\sigma'}x^\sigma$ für jedes $a \in R$, $x \in A$, wo σ' ein Automorphismus von R ist, und gilt ferner für irgendzwei Elemente $P, Q \in R$ mit $PQ=1$ auch $QP=1$, dann lässt sich für alle Automorphismen von R_n eine normale Darstellung gewinnen mit Hilfe der Automorphismen von R und der Elemente von R_n, die eine Inverse besitzen.

R. Moufang (Frankfurt a.M.).

André, Johannes. Über Perspektivitäten in endlichen projektiven Ebenen. Arch. Math. **6**, 29–32 (1954).

A perspectivity in a projective plane is a collineation which fixes every point of some line, which may be taken as the line at infinity. It will also fix all lines through some point. If this point is on the line at infinity, we call the perspectivity a translation. The perspectivities in a finite plane will have the translations as a normal subgroup. We may also regard the perspectivities fixing points at infinity as collineations of an affine plane, taking lines into parallel lines. Several results on groups of perspectivities are obtained here. Thus the translations are transitive on the finite points if and only if the group of perspectivities is transitive on the finite points. More striking is the result that if a group of perspectivities is intransitive on the finite points, then for only one of the transitive constituents can the subgroup fixing a finite point be different from the identity. *Marshall Hall, Jr.* (Columbus, Ohio).

Neumann, Hanna. On some finite non-desarguesian planes. Arch. Math. **6**, 36–40 (1954).

It is shown here that a large class of non-Desarguesian planes contain subplanes with seven points, the Fano configuration of a quadrilateral with collinear diagonal points. More exactly, by ingenious use of a construction due to the reviewer, she has shown that for every odd prime p and every $q=p^n$, there exist projective planes of q^4+q^2+1 points containing both a Desarguesian subplane of q^2+q+1 points and a subplane of 7 points. Similarly, with $q=2^n\geqq 4$ there exist projective planes of $2^{4n}+2^{2n}+1$ points containing both a Desarguesian subplane of $2^{2n}+2^n+1$ points and a quadrangle whose diagonal points are not collinear. This last result contains as a special case the result for $q=4$ due to H. Lenz [Arch. Math. **4**, 327–330 (1953); MR **15**, 461].

Marshall Hall, Jr. (Columbus, Ohio).

Klingenberg, Wilhelm. Euklidische Ebenen mit Nachbarelementen. Math. Z. **61**, 1–25 (1954).

In the real Euclidean plane, each line determines a reflection about itself. A reflection is an involution (i.e. of order two). The reflections about two different lines permute if and only if the lines are perpendicular, and in this case the product of the linear reflections is the reflection through their point of intersection. Bachmann has used these properties as a basis for a set of axioms for Euclidean geometry [Grundlagen der Geometrie, Kiel, 1951], defining a plane in terms of a group generated by involutions satisfying a number of requirements. To these axioms the author joins the idea of neighboring elements due to Hjelmslev [Danske Vid. Selsk. Mat.-Fys. Medd. **8**, no. 11; **10**, no. 1 (1929); **19**, no. 12 (1942); **22**, nos. 6, 13 (1945); **25**, no. 10 (1949); MR **7**, 472, 473; **8**, 83; **11**, 124]. He uses a set of eleven axioms on points, lines, perpendicularity, and neighboring elements. These axioms lead to the Theorem of Pappus, and from an earlier paper of his [Math. Z. **60**, 384–406 (1954); MR **16**, 507], he can introduce coordinates from an H-ring. This is a commutative ring in which elements either have inverses or are divisors of zero. Of any two divisors of zero, one must be a multiple of the other. Points are pairs (x, y) and lines are given by $ux+vy+w=0$. Lines $u_ix+v_iy+w_i=0$, $i=1, 2$, are perpendicular if $u_1u_2+kv_1v_2=0$ where u^2+kv^2 is a definite quadratic form in the sense that it cannot be zero unless both u and v are divisors of zero.

Marshall Hall, Jr. (Columbus, Ohio).

Convex Domains, Extremal Problems, Integral Geometry

Coxeter, H. S. M. On Laves' graph of girth ten. Canad. J. Math. **7**, 18–23 (1955).

F. Laves [Z. Kristallogr., Mineral. Petrogr. Abt. A. **82**, 1–14 (1932)] discovered a packing of equal spheres of density $\pi/12\sqrt{2}$ in which each sphere touches three other spheres. The graph obtained when the centres of the spheres are taken as the vertices, and the centre of each sphere is joined by an edge to the centres of the spheres which touch it, is here called Laves' graph. It has degree 3 and girth 10.

In this paper Laves' graph is derived from the group $S_1^2S_2S_1^2=S_2$, $S_2^2S_1S_2^2=S_1$ by Frucht's method [same J. 7, 8–17 (1955); MR 16, 671]. It is also shewn how the graph can be inscribed in an infinite regular skew polyhedron with square faces, six at each vertex. The author also describes a finite graph of degree 3 with $8lmn$ vertices and $12lmn$ edges, which has girth 10 provided l, m, n are all greater than 2. If $l=2$ the graph has girth 8, if $l=m=n=2$ the graph can be embedded in a surface to form a regular map of 24 octagons. (The special case $l=m=n$ is discussed by Frucht (loc. cit.).) *G. A. Dirac* (Vienna).

Dulmage, Lloyd. Tangents to ovals with two equichordal points. Trans. Roy. Soc. Canada. Sect. III. (3) **48**, 7–10 (1954).

The existence or non-existence of ovals with two equichordal points has been discussed by W. Süss [Tôhoku Math. J. **25**, 86–98 (1925)] and the reviewer [J. London Math. Soc. **27**, 429–437 (1952); MR **14**, 309]. The new contribution in the present paper to this problem consists of establishing a property which all the tangents to such a curve must have if the curve is to exist. *G. A. Dirac.*

Ehrhart, Eugène. Sur les ovales et les ovoïdes. C. R. Acad. Sci. Paris **240**, 583–585 (1955).

This note is mainly concerned with non-central convex bodies. Methods similar to those developed by the author in two previous articles [Rev. Math. Spéc. **64**, 61–62, 85–87 (1953); C. R. Acad. Sci. Paris **240**, 483–485 (1955); MR **16**, 574] here lead to results both geometrical and arithmetical in character. Among them is the following analogue of Minkowski's fundamental theorem on central convex bodies. Theorem. Let K denote a closed convex solid of revolution in 3-dimensional space. Let Λ denote any lattice of determinant 1 having a point at the centroid G of K. Then, if $V(K)\geqq 4^4/3^3$, K contains at least one point of Λ, other than G. *J. H. H. Chalk* (London).

Algebraic Geometry

Have

★Primrose, E. J. F. Plane algebraic curves. Macmillan & Co., Ltd, London; St. Martin's Press, New York, 1955. vii+111 pp. $3.00.

This is an elementary text book designed to give an undergraduate student a brief introduction to the theory of plane algebraic curves. The fundamental ideas are illustrated by well chosen examples worked out in detail. The topics covered are: curve tracing in the real Euclidean plane; line equations; quadratic transformations; intersections of two curves; Plücker's equations; cubic curves; and the genus of a curve. *G. B. Huff* (Athens, Ga.).

Selmer, Ernst S. The exceptional points of a cubic curve which is symmetric in the homogeneous variables. Math. Scand. **2**, 227–236 (1954).

The present paper deals with symmetric cubic elliptic curves

$$(1)\quad a(x+y+z)^3+b(xy+xz+yz)(x+y+z)+cxyz=0,$$

where a, b, c are rational integers satisfying the inequality

$$c(27a+9b+c)(b^3+b^2c-ac^2)\neq 0;$$

these are precisely the cubic elliptic curves which possess three rational inflections: $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$. In the general case when $c\neq -3b$, i.e. if the tangents at these three points are non-concurrent and the cubic is not equianharmonic, by a convenient choice of coordinates the equation of the cubic can be written in the reduced form, (2) say, deducible from (1) by assuming $b=0$.

Using a method due to A. Hurwitz [Vierteljschr. Naturforsch. Ges. Zürich **62**, 207–229 (1917)], for any rational point (x, y, z) of the curve (2)—where x, y, z are integers with no common factor—the weight $|xyz|$ is considered, and this number is then compared to the weight of the tangential point of (x, y, z). Hence it is proved that, if the tangential point is not an inflection, its weight is never smaller than the weight of the original point; moreover, the cases when the two weights can be equal are investigated. This leads to the possibility of studying the exceptional points of the curves (1) or (2), and to the determination of those among these curves which have an exceptional subgroup of order nine (cyclic in a real field), six or twelve. Among other related results it is also shown that, when a is squarefree, the number of exceptional points of the curve (2) can only be three (the inflections) or six. *B. Segre* (Rome).

Thalberg, Olaf M. 'Conic involutions' with a coincident curve of order $4n+2$. Avh. Norske Vid. Akad. Oslo. I. **1953**, no. 2, 13 pp. (1954).

Reviews of two previous papers, by the same author, on conic involutions have already appeared [same Avh. **1947**, no. 1; **1952**, no. 1; MR **9**, 464; **16**, 65]. In the present paper the general case considered is that of a plane involution of order 2 for which, as usual, the conics through four fixed points A, B, C, D are all invariant, while the coincidence curve is of the form C^{4n+2} $(A^{2n+1}, B^{2n+1}, C^{2n}, D^{2n})$. The case $n=1$ is considered in very great detail. The author maintains in this paper his silence about any possible value of a preliminary quadratic transformation with A, B, C as fundamental points. *J. G. Semple* (London).

Fava, Franco. Contributi allo studio della riflessione rispetto ad una curva. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 225–241 (1954).

New proofs and extensions of some of the theorems developed by the late E. Kasner in the study of the Schwarzian reflection $R(I, J)$ with respect to a given algebraic curve $\bar{C}$ of degree $\bar{n}$ and class $\bar{m}$ in the extended imaginary Euclidean plane are given. It is noted that I and J can be two arbitrary distinct points at infinity. If I and J are the circular points at infinity, then $R(I, J)$ reduces to what is ordinarily called Schwarzian reflection. Under $R(I, J)$, a single point P is transformed into $\bar{n}^2$ points $P'_{i,k}$. If I and J are the points at infinity on the coordinate axes, then $R(I, J)$ is algebraic and of the form: $X=f(y)$, $Y=g(x)$. In general, an algebraic curve C of degree n and class m is carried by $R(I, J)$ into an algebraic curve C' of degree $\bar{n}^2n$ and class $(2\bar{m}n+m\bar{n})\bar{n}$. If C coincides with $\bar{C}$, then C' is reducible and consists of $\bar{C}$ and another algebraic curve $\bar{S}$, called the satellite of C by Kasner. The degree of S is $\bar{n}(\bar{n}-1)^2$, and its class is $\bar{m}(\bar{n}-1)(3\bar{n}-5)$. The foci of C are also foci of S. This theory is applied when C is a conic section. Also, special degenerate cases are studied. *J. De Cicco.*

Brusotti, Luigi. Fasci reali di curve algebriche a curva reale generica massimale. Rend. Mat. e Appl. (5) **14**, 239–251 (1954).

Let Σ be the real part of an algebraic surface which carries a real pencil Φ of real curves of genus p which are maximal

in the sense that their real parts consist of $p+1$ circuits. If suitable restrictions are placed on the base points and critical centers of Φ, the set $\bar{\Sigma}$ will decompose into m $(1 \leq m \leq p+1)$ components, each of which is one of four possible types which are topologically distinct. The author examines the possible combinations of types which can occur in the case of rational pencils Φ_0 and gives several examples.

H. T. Muhly (Iowa City, Iowa).

Gudkov, D. A. On the space of coefficients of plane algebraic curves of the nth order. Dokl. Akad. Nauk SSSR (N.S.) 98, 337–340 (1954). (Russian)

Let C be a curve of (fixed) degree n in the projective plane (x, y, z) over the field of complex numbers. The relation $C(A)=0$ expresses that C contains the point A. The points $A, B, \cdots$ are said to be independent if the conditions $C(A)=0$, $C(B)=0, \cdots$, for the coefficients of C are linearly independent. C is called normal and denoted by C_k if its only singular points are k nodes $D_1, \cdots, D_k$. Furthermore, $D_1, \cdots, D_p$ are called independently variable if, taking $\bar{D}_1, \cdots, \bar{D}_p$ arbitrarily in the neighbourhoods of $D_1, \cdots, D_p$ resp., we can find $\bar{D}_{p+1}, \cdots, \bar{D}_k$ in the neighbourhoods of $D_{p+1}, \cdots, D_k$ such that there exists a $\bar{C}_k$ whose nodes are $\bar{D}_1, \cdots, \bar{D}_k$. The author gives eight theorems of which we mention the following ones. (i) Let C_k be real. Then in each of the cases $k \leq 2n+1$, $k=\frac{1}{2}n(n-3)$, and $k=\frac{1}{2}(n-1)(n-2)$ the nodes of C_k are independent. (ii) Let C_k be real, and let C_{kx}, C_{ky}, C_{kz} denote partial derivatives. Then the nodes $D_1, \cdots, D_p$ of C_k are independently variable if $2p+k<\frac{1}{2}n(n+3)$ and if the system

$$C_{kx}(D_i)=C_{ky}(D_i)=C_{kz}(D_i)=0 \quad (i=1, \cdots, p),$$
$$C_k(D_i)=0 \quad (i=p+1, \cdots, k)$$

is linearly independent in the coefficients of C_k. (iii) Let C be real, normal and rational. Then if all singular points of C are real, any number p of them with $p<\frac{1}{2}(3n-1)$ is independently variable. *F. J. Terpstra* (Pretoria).

Gudkov, D. A. The complete topological classification of nonsingular real algebraic curves of the 6th order in the real projective plane. Dokl. Akad. Nauk SSSR (N.S.) 89, 521–524 (1954). (Russian)

Algebraic curves in the real projective plane are said to be topologically equivalent when there exists a topological transformation of the plane into itself which defines a 1-1 correspondence between the points of both curves. According to a theorem of Harnack [Math. Ann. 10, 189–198 (1876)] no more than 68 distinct types of curves of the 6th order are possible. I. Petrowsky [Ann. of Math. (2) 39, 189–209 (1938)] proved the hypothesis of Hilbert [Math. Ann. 38, 115–138 (1891)] that the type consisting of 11 ovals lying outside each other does not exist. In the present paper all 68 types are considered and it is shown that only 53 of them do exist. *F. J. Terpstra* (Pretoria).

Turri, Tullio. Le trasformazioni birazionali involutorie in S_3 aventi una stella unita di rette. Rend. Sem. Fac. Sci. Univ. Cagliari 24, 28–35 (1954).

Turri, Tullio. Trasformazioni birazionali involutorie dello spazio associate a un complesso lineare. Rend. Sem. Fac. Sci. Univ. Cagliari 24, 36–45 (1954).

Scafati, Maria. Sulle superficie ellittiche con un fascio ellittico di curve di genere quattro. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 721–724 (1954).

Si tratta di una nota riassuntiva, in cui si dà notizia di alcuni risultati relativi alla classificazione, in tipi birazionalmente distinti, delle superficie ellittiche con un fascio ellittico, $\{C\}$, di curve di genere $\pi=4$. Seguendo un metodo usato per la prima volta da Enriques, si determinano dapprima le soluzioni compatibili con le caratteristiche aritmetiche delle superficie ellittiche in esame. Il problema di provare o escludere l'esistenza delle soluzioni così trovate viene poi ricondotto a quello di classificare le curve di genere quattro, che posseggono gruppi di trasformazioni birazionali in sè, ciclici o abeliani a base due. In questa nota vengono enunciati i risultati di tale disegno nel caso in cui le curve C, di genere quattro, sono iperellittiche. *F. Gherardelli.*

Gutwirth, Azriel. Transformations birationnelles planes de période deux. Rend. Mat. e Appl. (5) 12 (1953), 347–359 (1954).

L'Auteur reprend la classification des transformations crémoniennes planes involutives donnée par Bertini [Ann. Mat. Pura Appl. (2) 8, 11–23 (1877)] en examinant les systèmes linéaires de courbes rationnelles ou elliptiques invariantes: il démontre en effet dans un premier théorème que dans toute transformation crémonienne plane involutive il y a au moins un faisceau de courbes, rationnelles ou elliptiques, invariantes.

A. Les transformations dans lesquelles il n'y a pas de faisceau de courbes rationnelles invariantes forment deux classes: lorsqu'elles présentent un faisceau unique de courbes elliptiques invariantes, elles appartiennent à la classe des involutions de Bertini. Lorsqu'elles admettent un réseau de courbes elliptiques invariantes, elles appartiennent à la classe des involutions de Geiser.

B. Les transformations dans lesquelles il n'y a aucun faisceau de courbes elliptiques invariantes forment diverses classes: lorsqu'elles admettent un faisceau unique de courbes rationnelles invariantes, elles appartiennent à une classe définie par une transformation de Jonquières d'ordre $n=4$ (pour $n=5$, non spéciale, voir C). Lorsqu'elles admettent plusieurs faisceaux de courbes rationnelles invariantes elles appartiennent à la classe des transformations de Jonquières d'ordre 3.

C. Les transformations qui admettent à la fois des faisceaux de courbes rationnelles, et elliptiques, invariantes, forment plusieurs classes qui peuvent être considérées comme limites communes aux cas A et B. Lorsqu'il y a à la fois un seul faisceau de courbes rationnelles invariantes, et un seul faisceau de courbes elliptiques invariantes, la transformation appartient à la classe des transformations de Jonquières d'ordre 5 spéciales, c'est-à-dire dont les points fondamentaux sont points-bases d'un faisceau de cubiques. Lorsqu'il y a un faisceau de courbes rationnelles invariantes et un réseau de courbes elliptiques invariantes, la transformation appartient à la classe des transformations de Jonquières d'ordre 4. Enfin, lorsqu'il y a à la fois des réseaux de courbes rationnelles, et elliptiques, invariantes, la transformation appartient à la classe de l'homologie harmonique.

L. Gauthier (Nancy).

Please see the note on p. 1337.

Gröbner, W. Die birationalen Transformationen der Polynomideale. Monatsh. Math. 58, 266–286 (1954).

The author points out the following three main results of his paper. i) Every birational transformation is the product

of a correspondence of Veronese, a projectivity and a projection. As a consequence he obtains: ii) The quotients $g_0, g_1^2/g_2, \cdots, g_1g_{d-1}/g_d$ (see below) of the coefficients of the Hilbert function are invariant under biregular transformations. iii) The resolution (Auflösung) of an arbitrary singularity of an algebraic variety. To obtain i), the author introduces the classical definition of a birational transformation: (1) $\rho\eta_i=\varphi_i(\xi_i)$ $(i=0, 1, \cdots, m)$, $\rho\xi_j=\psi_j(\eta_j)$ $(j=0, \cdots, n)$, in the following fashion. Let K be an arbitrary field, and $x_0, \cdots, x_n$ and $y_0, \cdots, y_m$ two sets of indeterminates over K. A rational transformation of the ideals of

$$K[x]=K[x_0, \cdots, x_n]$$

onto the ideals of $K[y]=K[y_0, \cdots, y_m]$ is produced by a substitution (2) $y_i\rightarrow\varphi_i(x)$, $\varphi_i \in K[x]$ $(i=0, 1, \cdots, m)$, where the $\varphi_i(x)$ are forms of the same degree, and such that a homogeneous ideal $\mathfrak{a}_x$ of $K[x]$ is mapped onto the homogeneous ideal $\mathfrak{a}_y$ generated by all the forms $\Phi(y)$ such that $\Phi(\varphi(x)) \in \mathfrak{a}_x$. The transformation (2) is called birational for the prime ideal $\mathfrak{p}_x$, if there is an inverse transformation (3) $x_j\rightarrow\psi_j(y)$, $\psi_j \in K[y]$ $(j=0, \cdots, n)$, where the ψ_j are forms of the same degree such that (4) $\psi_j(\varphi(x))\equiv x_jM(x)$ $(\mathfrak{p}_x)$, $M(x)\not\equiv 0$ $(\mathfrak{p}_x)$ $(j=0, \cdots, n)$. The author calls every birational correspondence (2) for which the φ_i are linear forms and $m<n$ a projection. Furthermore, a projection is called regular if it is one-to-one for every subvariety. The author gives some immediate consequences of these definitions, such as: a prime ideal and its transform by a birational correspondence have isomorphic residue fields. If P is a simple point [in the sense of Zariski, Amer. J. Math. **61**, 249–294 (1939), p. 251] of $\mathfrak{p}_x$, and $\mathfrak{p}_y$ is the transform of $\mathfrak{p}_x$ by a birational correspondence, (2), (3), and $M(x)\not\equiv 0$ over P, then the transform of P is also a simple point of $\mathfrak{p}_y$. He then discusses how the Hilbert's function for regular subvarieties varies under the above transformations and the Veronese transformation, and states that if

$$H(t; \mathfrak{p}_x)=g_0+g_1t+\cdots+g_dt^d,$$

and $\mathfrak{p}_x$ and $\mathfrak{p}_y$ are regular birational equivalent prime ideals (in a certain sense), then ii) is valid. If $\mathfrak{a}_x=(a_0, \cdots, a_s)$ is the ideal of any singularity of a variety represented by $\mathfrak{p}_x$, then the singularity represented by $\mathfrak{a}_x$ is resolved (aufgelöst) by the transformation (5)

$$y_{ik}\rightarrow x_ia_k(x) \quad (i=0, \cdots, n;\ k=0, \cdots, s),$$

since, as the author states, "ihre Punkte verschwinden, d.h. haben keine Bildpunkte".

The reviewer doesn't see any advantage in employing the ambient space instead of the proper varieties for the study of the birational correspondences. Also, he must point out that definition (2) is equivalent to taking $y_i=\varphi_i(x)$, and considering $K[y]$ as a subring of $K[x]$, since it is verified that $\mathfrak{a}_y=\mathfrak{a}_x\cap K[y]$. This definition is not generally equivalent to the ordinary definition of the rational correspondences, since every algebraic correspondence is defined by a system of bihomogeneous equations in two sets of indeterminates, and it is verified that in every case the ideal (6) $K[x](\varphi_0, \cdots, \varphi_m)$ is not irrelevant, but corresponds by (2) to an irrelevant ideal of $K[y]$, whilst in the complete definition it always corresponds to (6), an ideal which is not irrelevant. Hence, in order to use definition (1), or definition (2) of the author, it is necessary to complete it by means of valuation theory [see Zariski, Trans. Amer. Math. Soc. **53**, 490–542 (1943); MR **5**, 11; Abellanas, Rev. Mat. Hisp.-Amer. (4) **9**, 175–233 (1949); MR **12**, 740], or by means of specialization [v. d. Waerden, Math. Ann. **110**, 134–160 (1934); A. Weil, Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; MR **9**, 303], or by a limit process as in the classical fashion, and as initiated in the author's last paragraph. For this reason, the author finds that the transformation (5) doesn't give any image for the points represented by $\mathfrak{a}_x$; but in the complete transformation defined by (5), it is verified that if $\mathfrak{p}_x$ is the ideal of an irreducible variety that contains the subvariety $\mathfrak{a}_x$ and $\mathfrak{p}_y$ is its transform by (5), and if (ξ) and (η) are homologous general points of $\mathfrak{p}_x$, and $\mathfrak{p}_y$, respectively, then it follows that $\mathfrak{m}_\eta=(a_0(\eta), \cdots, a_s(\eta))$ is the transform by (5) of $\mathfrak{a}_\xi=(a_0(\xi), \cdots, a_s(\xi))$, and that if $\mathfrak{a}_\xi$ is a prime ideal, then the homogeneous quotient rings of $\mathfrak{a}_\xi$ and $\mathfrak{a}_\eta$ are equal to one another. Hence the transformation (5) doesn't signify any progress for the resolution of the singularities.

P. Abellanas (Madrid).

Marchionna, Ermanno. Sull'identità birazionale delle ipersuperficie multiple diramate da una medesima varietà. Ann. Mat. Pura Appl. (4) **37**, 265–290 (1954).

A_μ is a "general" primal of order μ in S_r, projected from a "general" point U_∞ into a μ-ple S_{r-1}; F_m is a "general" primal of order m in S_{r-1}, on which the μ-ple ambient determines a μ-ple F_m, denoted by F_u (u being the μ-valued algebraic irrationality) with the branch locus D, intersection of F_m with the branch locus in the ambient.

The three words "general" here are not defined, but from what follows they appear to imply that A_μ and F_m are non-singular, that F_m has no contact with the branch locus of the multiple S_{r-1}, and that on the μ-ple surface, general S_3 section of the μ-ple F_m, the branch curve has the expected number of essential cusps (branching of three sheets) and essential nodes (branching by pairs of four sheets) and no other singularity.

The theorem proved is: If $\mu>4$, any ν-ple locus $F_{u'}$ covering F_m and having D as its branch locus is birationally equivalent to the general μ-ple F_m defined above, and $\mu=\nu$.

The proof involves induction over r, as it is shewn that if the theorem is true for the general prime sections of F_u, $F_{u'}$ it is true for F_u, $F_{u'}$ themselves. For $r=4$, 3 the general μ-ple surface and μ-ple curve, its plane section, are projected into an $m\mu$-ple plane and line, and it is shewn by a degenerate-limit method that on the Riemann surface of the multiple line the $m\mu$ sheets can be so named and circuits round each branch point so chosen that the sheets are interchanged in a quite determinate manner, which is stated in detail. For $\mu>4$, exactly the same can be done for the $m\nu$-ple line obtained in the same way from $F_{u'}$, whence $\mu=\nu$ and the two multiple lines are birationally equivalent.

P. Du Val (London).

Rosati, Mario. Sull'equivalenza birazionale delle due varietà di Picard associate ad una varietà superficialmente irregolare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 708–715 (1954).

Let W be an algebraic variety of irregularity $p>0$, V the Abelian variety determined by the periods of the Abelian integrals of the first kind on W, and let V' be the dual Abelian variety of V defined in the sense of Castellnuovo. Since it is possible to define V' on V, the author takes directly the variety V. Following Conforto [Funzioni abeliane modulari, v. 1, Edizioni Univ. "Docet", Roma, 1951; MR **14**, 859], the author finds that a necessary and sufficient condition for the birational equivalence of V and V' is that $\omega M\omega'=0$, where ω is an arbitrary Riemann matrix associated to V and M an integer matrix with

$|M|=1$. This condition is an immediate consequence of G. Scorza's condition: $\alpha\bar{\omega}=\omega A$ [see Conforto, loc. cit., p. 46]. The author also gives a classification of the Abelian varieties in that are birationally equivalent to their dual Picard varieties with respect to the birational equivalence.

P. Abellanas (Madrid).

Gallarati, Dionisio. Un'osservazione sul massimo numero di punti doppi delle superficie algebriche. Atti Accad. Ligure 8 (1951), 353–355 (1952).

F. Severi a proposé [Ann. Mat. Pura Appl. (4) 25, 1–41 (1946); MR 9, 609] comme maximum du nombre de points doubles isolés présentés par une surface algébrique F^m de S_3 l'expression: $d=\binom{m+2}{3}-4$. B. Segre a ensuite montré [Boll. Un. Mat. Ital. (3) 2, 204–212 (1947); MR 9, 608] que ce résultat est incorrect: il existe des surfaces présentant un nombre de points doubles isolés supérieur à d.

L'Auteur montre ici que les surfaces d'équations:

$$F^{2n}=\prod_{i=1}^{2n} L_i+P_n^2=0, \quad F^{2n+1}=\prod_{i=1}^{2n+1} L_i+P_n^2L_{2n+2}=0,$$

où les L_i sont des formes linéaires et P_n une forme de degré n, présentent respectivement $n^2(2n-1)$ et $n^2(2n+1)$ points doubles isolés, aux points $P_n=L_i=L_j=0$ $(i\neq j\neq 2n+2)$. Ce nombre dépasse d pour $m=2n\geqq 10$, et $m=2n+1\geqq 13$. Comme la limite d est correcte pour $m=4$ (surface de Kummer) il reste seulement à examiner sa validité pour les cas $5\leqq m\leqq 9$ et $m=11$.

L. Gauthier (Nancy).

Nisnevič, L. B. On the number of points of an algebraic manifold in a prime finite field. Dokl. Akad. Nauk SSSR (N.S.) 99, 17–20 (1954). (Russian)

The author considers an absolutely irreducible variety V in n-dimensional projective space; the field of definition of V is k and has characteristic p; dim $V=d$; degree $V=m$; N is the number of points V whose coordinates are in the prime field of k. It is shown that

$$|N-p^d|\leqq cp^{d-\frac{1}{2}},$$

where c depends on m and n but not on p or d. The proof is by induction on d and uses the well-known case $d=1$ studied by A. Weil. It should be observed that in Th. 3 seemingly both h and R denote dim H.

F. J. Terpstra.

Chow, Wei-Liang. Abelian varieties over function fields. Trans. Amer. Math. Soc. 78, 253–275 (1955).

The paper begins by the proof of some auxiliary results which are of interest by themselves. If an abelian variety A is defined over k, then every algebraic subgroup X of A is normally algebraic over the algebraic closure $\bar{k}$ of k. If, furthermore, X is a variety (necessarily abelian) defined over a primary extension K of k (i.e. an extension K such that $K\cap\bar{k}$ is purely inseparable over k), then X is defined over k. If an abelian variety A is defined over a regular extension $k(u)$ of k, and if s is a k-isomorphism such that $k(u)$ and $k(u^s)$ are linearly disjoint over k and that A^s is biregularly isomorphic to A, then A is biregularly isomorphic to some abelian variety defined over k.

The author now considers an algebraic family $(\mathfrak{F})$, defined over k, of abelian varieties. Its generic member A^0 is an abelian variety defined over a regular extension K^0 of k, and almost all members of $(\mathfrak{F})$ are abelian varieties. Then there exists an abelian variety A defined over k and a rational homomorphism F (defined over K^0) of A^0 onto A which are universal for the rational homomorphisms of A^0 into abelian varieties defined over separable extensions of k which are linearly disjoint from K^0. This essentially unique variety is called the k-image of A^0 over K^0. It is obtained by the usual "product construction" of universal mappings, the results of a previous note [Chow, Proc. Nat. Acad. Sci. U. S. A. 38, 1039–1044 (1952); MR 14, 580] permitting one to avoid infinite products. The kernel of F is an abelian subvariety of A^0. If A^0 is a rational image of some abelian variety defined over an extension of k which is linearly disjoint from K^0, then F is one-to-one. The notion of k-image is used for strengthening some results about quotient abelian varieties [loc. cit.].

In a dual way there exists an abelian variety A' defined over k and a rational isomorphism F' (defined over K^0) of A' into A^0 which are universal for the rational homomorphisms into A^0 of abelian varieties defined over extensions of k which are linearly disjoint from K^0. It is called the k-trace of A^0 over K^0. The k-trace A' is isogeneous to the k-image A of A^0, and $F'(A')$ is the only abelian subvariety of A^0 which is isogeneous to A. The k-trace A' of A^0 and the mapping F' remain unchanged if K^0 is replaced by a regular extension of K^0. The k-image A and the mapping F remain unchanged if K^0 is replaced by a primary extension of K^0, but may change under non-primary extension of K^0.

P. Samuel (Cambridge, Mass.).

Matsusaka, Teruhisa. A note on my paper, "Some theorems on Abelian varieties." Nat. Sci. Rep. Ochanomizu Univ. 5, 21–23 (1954).

Construction of a projective "derived normal model" U of a projective variety W, in the case in which the function field of U is a given finite algebraic extension of the function field of W.

P. Samuel (Cambridge, Mass.).

Morikawa, Hisasi. Cycles and endomorphisms of abelian varieties. Nagoya Math. J. 7, 95–102 (1954).

In the notation of Weil's "Variétés abéliennes et courbes algébriques" [Hermann, Paris, 1948; MR 10, 621], if X and Y are cycles of complementary dimensions on an abelian variety A, and if t is generic for A over a field of definition k for A, X, and Y, one has $S[X\cdot Y_t]=\delta(X, Y)t+a$, where $a\in A$ is rational over k and $\delta(X, Y)$ is an endomorphism of A. The endomorphism $\delta(X, Y)$ depends only on the classes of algebraic equivalence of X and Y and satisfies certain simple identities. If A^n is a jacobian variety and $1\leqq r<n$, then $\delta(W^{(n-r)}, X^r)$ is symmetric and a suitable scalar multiple of any symmetric endomorphism of A is of this form (for any given r). Some of the arguments are unclear, and there are numerous typographical errors, the most serious being the omission of an exclamation point in proposition 9.

M. Rosenlicht (Rome).

Roth, Leonard. Para-Abelian varieties. Rend. Mat. e Appl. (5) 14, 30–37 (1954).

L'Autore considera una varietà algebrica W_p, di dimensione p, sulla quale esista una congruenza $\{V_q\}$ $(1\leqq q\leqq p-1)$ di varietà di Picard, i cui elementi irriducibili siano non singolari e birazionalmente equivalenti (ma che può contenere un certo aggregato di elementi riducibili di vario tipo) e prova che W_p possiede una seconda congruenza $\{V_{p-q}\}$ di varietà V_{p-q} d-secanti le V_q. W_p è una varietà para-abeliana del tipo q se può essere rappresentata sul prodotto d-plo $W_p^*=V_q^*\times V^*_{p-q}$ con V_q^* e V^*_{p-q} immagini di $\{V_{p-q}\}$ e $\{V_q\}$ rispettivamente, in guisa che il luogo delle coincidenze della rappresentazione consista interamente di varietà generate da elementi riducibili di $\{V_q\}$

e $\{V_{p-q}\}$, e che non ci siano altre eccezioni nella corrispondenza. L'Autore dimostra che i sistemi canonici $X_k(W_p)$ $(k=p-q, p-q+1, \cdots, p-1)$ appartengono alla congruenza $\{V_q\}$ oppure hanno ordine zero, e trova che l'irregolarità superficiale q_2 di W_p soddisfa la diseguaglianza $q_2 \geqq q_2'+q_2''$ ove q_2' e q_2'' sono le irregolarità di $\{V_q\}$ e $\{V_{p-q}\}$; inoltre costruisce un modello proiettivo di W_p qualora l'involuzione i_q che $\{V_{p-q}\}$ sega sulla generica V_q sia ciclica. *D. Gallarati* (Genova).

Differential Geometry

Hav'e

✱Julia, Gaston. Cours de géométrie infinitésimale. Troisième fascicule. Géométrie infinitésimale. Première partie: Méthodes générales. Théorie des courbes. Gauthier-Villars, Paris, 1955. 220 pp. 3500 francs.

This section 3[1] of Professor Julia's book [see MR 15, 352; 16, 512, 619] has chapters on the theory of contact, on envelopes, ruled surfaces and line complexes, as well as on the derivation and some applications of the Frenet formulas. The treatment of contact is enlivened by a discussion of contact transformations. Since the discussion of contact, envelopes and related topics is rather strictly in the Lagrange-Cauchy tradition and the few references usually go back to the books by Darboux or Goursat, there is something static about this treatise. But the information it brings is solid, and there is no doubt that the student of this book will be able to obtain a good working knowledge of the classical theories dealing with the application of analysis to geometry. *D. J. Struik* (Cambridge, Mass.).

Sauer, R. Elementargeometrische Modelle zur Differentialgeometrie. Elem. Math. **9**, 121–131 (1954); **10**, 4–11, 25–32 (1955).

Many theorems of differential geometry of surfaces, especially those depending only on the first fundamental form have elementary analogues in what might be called difference geometry. For this the surfaces are replaced by nets (Gitter) of line segments forming polygons corresponding to nets of curves on a surface. The openings in the net, which correspond to elements of area on a surface may be either triangular and hence necessarily planar, or in the form of a quadrilateral which may or may not be planar.

Using triangular nets analogues are developed for Gaussian curvature, geodesic curvature, and the Gauss-Bonnet Theorem. Several types of quadrilateral nets are considered, including the plane cornered net, the equiangular net, and the plane cornered Chebyshev net, and their properties are developed, especially their deformation properties. Results obtained give analogues of a number of well-known theorems in differential geometry. For example, consideration of the plane cornered Chebyshev nets leads to an analogue of Enneper's theorem on the torsion of asymptotic lines on surfaces of constant negative curvature.

It is shown that by a suitable limiting process, the results obtained go over into familiar theorems of differential geometry and the properties give considerable insight into the geometric meaning of these theorems.
S. B. Jackson (College Park, Md.).

Gándara, Alfonso Nápoles. Some theorems on the variation of the length of a variable curve sliding on a curved surface. Bol. Soc. Mat. Mexicana **8**, 47–50 (1951). (Spanish)

Pinl, M. Geschlossene Minimalflächen. Compositio Math. **12**, 178–184 (1954).

Consider the closed V_2 in Euclidean 4-space given by
$$x_1=\cos u,\quad x_2=\sin u,\quad x_3=\cos v,\quad x_4=\sin v.$$
It is shown that this V_2 is not a minimal surface in Euclidean 4-space, but that this V_2 is embedded in a V_3 defined by $x_1^2+x_2^2+x_3^2+x_4^2=2$, and that with respect to this V_3 it is a minimal surface. This illustrates a known theorem by Schouten and Struik [Einführung in die neueren Methoden der Differentialgeometrie, Bd. II, 2nd ed., Noordhoff, Groningen, 1938, p. 94]. It is noted that there are no spherical or hyperspherical minimal surfaces in a Euclidean R_n $(n \geqq 3)$ for which the discriminant of the first fundamental form does not vanish. *S. B. Jackson.*

Woinaroski, Rudolf. Les déplacements rigides dans un espace euclidien à trois dimensions. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **2**, 260–284 (1951). (Romanian. Russian and French summaries)

Certain results of this paper concern the geometry of ruled surfaces and are well known [cf., e.g., Hilbert and Cohn-Vossen, Anschauliche Geometrie, Springer, Berlin, 1932, p. 184]. Other results are purely geometric and pertain to orthogonal trajectories (transversals) of the generators of ruled surfaces, and are implicit in Blaschke, Vorlesungen über Differentialgeometrie, vol. I, 3rd ed., p. 272 [Springer, Berlin, 1924]. The kinematic results are based on certain relations between the velocity field and the geometry of the axodes (generalizing the Euler-Savary equations), derived in a previous paper [Disquisit. Math. Phys. **4**, 175–239 (1945); MR **8**, 532]. These relations are applied to several problems in which one or both axodes are to be determined from loci of points or lines of the moving body. It is not explained why arbitrarily chosen transversals (rather than the intrinsic lines of striction) are used to represent the axodes, or why no use of the extremely efficient Clifford (dual) numbers to represent straight lines [cf. Blasche, loc. cit.] is made. Finally, one misses a discussion of the nontrivial issue of the signs of the curvature and torsion of the transversals. *A. W. Wundheiler.*

Palman, Dominik. Die Flächen 3. Ordnung mit vier Doppelpunkten. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 129–150 (1954). (Serbo-Croatian summary)

Ist H_k eine hyperbolische Geradenkongruenz und Ψ eine Fläche zweiter Ordnung, so kann man durch einen beliebigen Punkt A des Raumes einen Strahl aus H_k legen und auf diesem den zu P bezüglich Ψ konjugierten Punkt bestimmen. Auf diese Weise erhält man für sämtliche Punkte A gewisse Bildpunkte. Diese Raumtransformation hat Verfasser in einer vorhergehenden Arbeit [vgl. D. Palman, Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **296**, 199–214 (1953)] kubische Inversion genannt und zahlreiche ihrer Eigenschaften studiert. Die Methode wird jetzt dazu benutzt, Flächen dritter Ordnung mit vier Doppelpunkten, die mittels kubischer Inversion erhalten werden, zu klassifizieren und zwar mit Rücksicht auf die Realität der Doppelpunkte und Geraden der Fläche. Von besonderem Interesse werden unter diesen Flächen diejenigen, welche durch den absoluten Kegelschnitt gehen und solche, von deren vier Doppelpunkten zwei in zwei Punkte des absoluten Kegelschnittes fallen. Durch zweckmässige Wahl der Leitgeraden der Kongruenz H_k und der Fläche Ψ so wie deren gegenseitige Lage erhält Verfasser sieben Arten

von Flächen dritter Ordnung mit vier Doppelpunkten mit Rücksicht auf die Relatität der Doppelpunkte und Geraden. Wenn die Fläche dritter Ordnung durch den absoluten Kegelschnitt geht, sind alle vier Doppelpunkte endliche Punkte, zwei davon sind immer konjugiert-imaginäre, während die übrigen auch reell oder zusammenfallend sein können. Auch die auf solchen Flächen liegenden Geraden werden ausführlich untersucht. Die in Rede stehenden Flächen tragen auch ∞^1 Kreise, die von vier Ebenenbüscheln ausgeschnitten werden. Zwei der Achsen dieser Büschel sind die Leitgeraden der Kongruenz. Zum Schluss wird der Fall betrachtet, wo Ψ eine Kugel mit reellem oder imaginärem Radius ist. *M. Pinl* (Köln).

Byušgens, S. S. On the theory of congruences of lines. Dokl. Akad. Nauk SSSR (N.S.) **97**, 381–384 (1954). (Russian)

A congruence in euclidean three-space is called bi-orthogonal if it can be generated as the curves of intersection of two families of orthogonal surfaces. Necessary and sufficient conditions are that at least one set of lines of curvature of the second order (bisectrices of the asymptotic lines) of the congruence is a normal congruence and in that case the second set is also normal. If these lines of curvature are undetermined, the congruence is called umbilical. Necessary and sufficient conditions for this case are given, and attention is paid to what happens when such a congruence is normal or consists of straight lines. *D. J. Struik.*

Frey, Annemarie, und Strubecker, Karl. Die Transformationstheorie der quadratischen Linienkomplexe [(11)(22)]. I. J. Reine Angew. Math. **193**, 209–238 (1954).

The quadratic line complexes K_p determined by the equation

$$p_{03}p_{12}=p(p^2_{01}+p^2_{02})\qquad p\neq 0$$

have Segre characteristic [(11)(22)]. This paper discusses the transformation theory of such complexes. Every complex of this type possesses a commutative continuous group G_3 of collinear automorphisms which forms the basis of its transformation theory. The transcendental mappings, which lead to central similarities, are obtained from the canonical representations of this G_3. *S. B. Jackson.*

Geldel'man, R. M. On the theory of a three-parameter complex of circles. Dokl. Akad. Nauk SSSR (N.S.) **99**, 201–204 (1954). (Russian)

The problem considered in this note is similar to that of line complexes in projective space. Using pentaspherical coordinates and referring the space to a Cartan frame S_0, $S_1, \cdots, S_4$, the infinitesimal displacement of the frame is given by $dS_\alpha=\omega_\alpha{}^\beta S_\beta$ $(\alpha, \beta=0, 1, \cdots, 4)$. Through the circle $[S_2S_3]$ there are three canal surfaces whose apheres are the focal spheres of the complex. The frame may be normalized so that these surfaces are $\omega^3=\omega_1{}^3=0$; $\omega^3=\omega^2=0$; $\omega_2{}^0-c_3{}^0\omega^2=c_2{}^0\omega_1{}^3+b_1{}^2\omega^2+c_1{}^2\omega^3$, while the focal spheres are

$$S_3;\quad S_3-\frac{1}{a_1{}^3}S_2;\quad S_3+\frac{b_2{}^0-c_0{}^0}{c_2{}^0}S_2.$$

The nature of the complex depends on the values of the constants involved above; thus if $b_2{}^0=0$ or $c_3{}^0=0$, S_4 describes a surface and all the circles of the complex are tangent to this surface; the complex then depends on two functions of 3 parameters. The other results obtained are of similar nature. *M. S. Knebelman* (Pullman, Wash.).

Glagoleva, N. N. On asymptotic transformations of a surface. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 151–168 (1954). (Russian)

Bianchi has characterized, in his "Lezioni di geometria differenziale" [v. 2, 3rd ed., Spoerri, Pisa, 1923, cap. XVIII], a W-congruence as a congruence for which the focal surfaces can be so mapped on each other that the asymptotic lines correspond. For such asymptotic transformations of surfaces some new theorems are derived with the aid of the method of exterior forms, as applied by Finikov to the theory of congruences. One of these theorems is as follows. Let a surface S allow a transformation into ∞^1 surfaces S' such that corresponding points P' of the S', lying in plane π tangent to S at M, lie on a circle with center at M, and the tangent planes at P' to the S' form the same angle with π, then S and S' are of the Bianchi form with asymptotic correspondence and have at correspondent points the same Gaussian curvature, for which already Bianchi has given the formula $K=-[f_1(u)+f_2(v)]^{-2}$, where $u=c_1$, $v=c_2$, are asymptotic lines. In another theorem this result is generalized to the affine case. *D. J. Struik.*

Havlíček, Karel. Canal W-surfaces. Časopis Pěst. Mat. **78**, 347–357 (1953). (Czech)

Soient $a_{\lambda\mu}$, $b_{\lambda\mu}$ les tenseurs fondamentaux de la surface et $b_{\omega\mu\lambda}$ le tenseur du 3ème ordre défini par l'équation

$$b_{\omega\mu\lambda}=\frac{\partial}{\partial\xi\omega}b_{\mu\lambda}-\left\{\begin{matrix}\alpha\\ \mu\omega\end{matrix}\right\}b_{\alpha\lambda}-\left\{\begin{matrix}\alpha\\ \lambda\omega\end{matrix}\right\}b_{\mu\alpha},$$

où α, λ, μ, $\omega=1, 2$, $\xi^1=u$, $\xi^2=v$. Pour les surfaces canales on a en paramètres principeaux l'équation connue

$$(1)\qquad b_{111}b_{222}=0.$$

Les surfaces canales qui sont les W-surfaces satisfont aussi à l'équation

$$(2)\qquad \frac{\partial R_1}{\partial u}\frac{\partial R_2}{\partial v}-\frac{\partial R_2}{\partial u}\frac{\partial R_1}{\partial v}=0,$$

où $R_1{}^{-1}$, $R_2{}^{-1}$ sont les courbures principales.

On a les théorèmes suivants. 1) Si la surface canale est la W-surface, alors elle est la surface de rotation ou l'enveloppe d'un système à un paramètre des sphères avec le diamètre constant. 2) La condition nécessaire et suffisante pour que la cyclide de Dupin soit W-surface est: la cyclide est un tore. L'auteur montre aussi une propriété caractéristique: sur les surfaces canales les courbes principales sont les courbes géodésiques. *F. Vyčichlo* (Prague).

Marussi, Antonio. Sulla curvatura tangenziale delle trasformate di curve nelle rappresentazioni affini fra superficie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 478–483 (1954).

Let there be an affine representation of the surface Σ on the surface Σ, it being understood that the two surfaces are referred to the same coordinates. When one studies Christoffel symbols of the second kind, a tensor $T_{rs}{}^i$ appears; one has

$$\overline{\left\{\begin{matrix}i\\ rs\end{matrix}\right\}}=\left\{\begin{matrix}i\\ rs\end{matrix}\right\}+T_{rs}{}^i.$$

If a curve (c) on Σ has geodesic curvature γ, then the corresponding curve on $\overline{\Sigma}$ has tangential curvature

$$\bar\gamma=\frac{S}{(m_i)^3}(\gamma+\epsilon_{ji}T_{rs}{}^i\lambda^r\lambda^s\lambda^j),$$

where λ^r represents the tangent to (c), m_t is the module of linear deformation in the direction λ^r, S is the module of area deformation, and ϵ_{ji} is the Ricci tensor for Σ. For conformal representations this formula reduces to the Theorem of Schols. *A. Schwartz* (New York, N. Y.).

Gambotto, Anna. Estensione della nozione di linee principali e determinazione delle V_3 aventi certe particolarità. Univ. e Politec. Torino. Rend. Sem. Mat. 13, 291–305 (1954).

The author defines the principal curves of a V_h in S_{2h+1} as the curves such that the tangent S_h's to the V_h in two infinitely near points on the curve are incident in a point with an order of approximation $\sigma \geqq 4$ (instead of $\sigma = 2$, as it happens for a generic curve on V_h). This definition extends the definition of the principal curves of a V_2 in S_5 given by Terracini. The principal directions through a point P of the V_h are generators of an algebraic cone Γ_{h-1}^{h+3} which has the point P as vertex.

After giving the classification of the V_3's for which the principal curves are undetermined, the author proves that, for the V_3's which are locus of ∞^1 planes, each generator plane is a five-fold component at least for the cone Γ_2^6; it is six-fold component if and only if the V_3 is a locus of ∞^1 planes each of which lies in an S_4 of a developable surface or meets the successive plane in a point. The author studies also, the V_3 locus of ∞^1 surfaces in S_3: in this situation the tangent plane to the surface is at least twofold component for the cone Γ_2^6 and it is at least threefold component when infinitely near S_3's have a point Q of intersection.

In the general case, i.e., if the order of approximation of two S_3's is $\sigma = 2$, the tangent plane to the surface is at least fourfold component if and only if the generator surface is a quadric through Q that is tangent in Q to a plane geometrically characterized by the S_3's. *C. Longo* (Rome).

Bompiani, E. Sugli elementi curvilinei piani E_3 tangenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 585–590 (1954).

The author gives a representation of the linear elements of third order, E_3's, of the plane with the same center and tangent, on the points of a quadric cone in a projective space S_3. He proves that the projective geometry of these E_3 is the geometry of the cone with respect to the group G_4 of the self-collineations of the cone which leave fixed two generators and a point O_3 of one of these. The author proves that two E_3's determine a covariant line; four E_3's have one invariant, not depending on their E_2's, which can be expressed as the cross-ratio of the tangent and three covariant lines.

The author proves also that two tangent E_3's and a line through the center allow one to introduce the notion of pencil of E_3's. These belong to a pencil of plane cubics and are mapped on the points of a plane section of the cone; special pencils correspond to planes through O_3 and belong to a pencil of conics. Finally, the author studies the E_3's with respect to point transformations and proves, among other results, that five elements are needed to obtain an invariant. *C. Longo* (Rome).

Özkan, Asim. Une condition caractéristique pour la classe des surfaces à courbure moyenne constante et un résultat pour ces surfaces. Arch. Math. 6, 136–138 (1955).

It was proved by Raffy [Bull. Soc. Math. France 20, 47–49 (1892)] that for the class of surfaces H=const., where H is the mean curvature, and all surfaces applicable to them there is a function of the Gaussian curvature K, namely $(H^2-K)^{1/2}$ such that the form $(H^2-K)^{1/2}ds^2$ has second curvature zero. It is shown here that conversely, if for a class of Weingarten surfaces and all surfaces applicable to them there is a function $F(K)$ so that $F(K)\,ds^2$ has zero second curvature for all surfaces of the class, then this class is the class H=const. It is also shown that two surfaces of constant mean curvature, H and H_1, cannot be applicable unless $H^2=H_1^2$. *S. B. Jackson* (College Park, Md.).

Pan, T. K. Complementary surfaces for a vector field. Proc. Amer. Math. Soc. 6, 151–158 (1955).

The author considers a vector field v on a surface S in ordinary space. A curve of v on S is defined as a curve to which the vectors v are tangent. The asymptotic lines of v on S are curves along which usual curvature of v is zero. The family of developable surfaces of v, which are formed by straight lines of v along the curves of v, have S as locus of edges of regression. In the same way there is a family of developable surfaces determined by the asymptotic lines of v. The locus of the edges of regression of this last family are studied in this paper. The center of geodesic curvature of an orthogonal trajectory of a geodesic family is included as a special case in the notion of orthocentre of associative curvature of v, and complementary surfaces for a geodesic family generalize into complementary surfaces for a vector field. *E. T. Davies* (Southampton).

Papy, Georges. Formes différentielles extérieures de classe C^1 sur une variété différentiable de classe C^1. Bull. Soc. Math. Belg. 6 (1953), 62–69 (1954).

Osservato che la nozione di forma differenziale esterna di classe C^r nell'intorno di un punto p di una varietà differenziabile V come forma di classe r in un certo sistema di coordinate locali non ha senso invariante (oltre che per le forme di grado 0), l'Autore propone una nuova definizione invariante di forme di classe C^1, il cui insieme costituisca un'algebra esterna, ciascuna delle quali si possa normalizzare in guisa da farla apparire come somma di forme monomie di classe C^1 rispetto ad una opportuna rappresentazione locale (non necessariamente la stessa per ogni monomio). Mediante tale normalizzazione si può sviluppare la teoria delle forme di classe C^1 in modo autonomo, ricorrendo soltanto alla teoria classica (e non alla teoria del differenziale generalizzato di P. Gillis, E. Cartan, B. Segre). L'Autore dimostra altresì che l'anello differenziale delle forme di classe C^1 è semplice. *D. Gallarati* (Genova).

Šulikovskiĭ, V. I. An invariant characteristic of the metric of a spiral surface. Dokl. Akad. Nauk SSSR (N.S.) 99, 35–36 (1954). (Russian)

A net on a surface is called a spiral net if it can be taken as the coordinate set of a line element $ds^2=e^{u+V}(du^2+dv^2)$, where $V=V(v)$. When a surface allows such nets it is called a spiral surface, a notation going back to Peterson. One invariant criterion for these nets is that a) the first and second Čebyšev vectors of the set are gradient vectors, and b) that the sum or difference of their potentials is harmonic. Criterion a) is satisfied for the so-called generalized spiral sets introduced by Ya. S. Dubnov [Trudy Sem. Vektor. Tenzor. Anal. 9, 7–48 (1952), esp. p. 26; MR 14, 1014]. In the present paper a second criterion is obtained by finding the consistency conditions for the system $p_{i|s}=h\epsilon_{si}-\sigma g_{si}$, where p_i is a properly normalized tangent vector to the lines of the net, h a constant, ϵ_{si} is the bivector ± 1, 0, and g_{si} is the metrical tensor, i, $s=1, 2$. These four equations must determine the three unknown functions p_i,

$\sigma=\sigma(u^1)$. Analysis of the integrability conditions gives that the case $K=0$ allows an infinite set of spiral nets, which for the plane consist of logarithmic spirals; the case $K=\text{constant}\neq 0$ allows only such nets if $h=0$; they can by bending be made into meridians and parallels. When $k\neq\text{constant}$ there are three types of spiral surfaces for which the invariant conditions are given in terms of K and its invariant derivatives. In the third of these cases the line element can be written in the form $ds^2=du^2+u^{-4c}dv^2$. On rotation surfaces of this type the spiral nets pass into each other by rotation around the axis. *D. J. Struik.*

Singh, Kamala Devi. Infinitesimal deformations in a subspace V_n of a Riemannian space V_m. Acad. Roy. Belg. Bull. Cl. Sci. (5) **40**, 1072–1079 (1954).

Considerata una V_n immersa in uno spazio di Riemann V_m si determinano le condizioni affinchè un vettore normale a V_n si deformi parallelamente (mantenendosi normale) in una deformazione infinitesima di V_n. Si studia il caso che V_n sia totalmente geodetica e che il vettore che determina la deformazione appartenga a V_n ed il caso che tutti i vettori tangenti in un punto P a V_n siano deformati parallelamente: in tal caso anche tutte le normali di V_n si deformano parallelamente. *C. Longo* (Roma).

Ôtsuki, Tominosuke. Isometric imbedding of Riemann manifolds in a Riemann manifold. J. Math. Soc. Japan **6**, 221–234 (1954).

Let M be a submanifold of dimension n in a complete Riemannian manifold V of dimension $n+N$. Let O be a fixed point of V and $P_0\in M$ be a point at which the geodesic distance from O to M attains a relative maximum or minimum. The author studies the behavior of M at P_0 and derives from it properties which depend only on the induced metric of M. In this way he finds lower bounds of the dimension of a complete Riemannian manifold in which a given Riemannian manifold can be isometrically imbedded. An important rôle is played by two arithmetic invariants, the index of relative nullity $\nu(P)$ and the index of nullity $\mu(P,K)$, which were first introduced by the reviewer and N. H. Kuiper in the case that V is the Euclidean space [Ann. of Math. (2) **56**, 422–430 (1952); MR **14**, 408]. Many theorems are proved, of which the following typical ones may be quoted: 1) If V is of non-positive curvature, there exists no compact minimal variety disjoint with the minimum point locus with respect to some point of V. 2) If V is of constant curvature K, then between the indices of nullity and relative nullity the inequalities $\nu(P)\leqq\mu(P,K)\leqq N+\nu(P)$ hold. 3) An n-dimensional Riemannian manifold of negative sectional curvature cannot be isometrically imbedded in a $(2n-2)$-dimensional Riemannian manifold of non-negative sectional curvature. *S. Chern* (Chicago, Ill.).

Mishra, R. S. Subspaces of a generalised Riemannian space. Acad. Roy. Belg. Bull. Cl. Sci. (5) **40**, 1058–1071 (1954).

A general Riemannian space is defined here as a space with a non-symmetric fundamental tensor $g_{\beta\alpha}$. It is well known that in such a space several connections can be defined. For one of these the formulae of Gauss and Codazzi for a subspace are developed and curves in such a subspace are considered. A space with a non-symmetric fundamental tensor can be looked upon as an arbitrary Riemannian space with the fundamental tensor $g_{(\beta\alpha)}$ and with a bivector-field $g_{[\beta\alpha]}$ defined in it. Thus all formulae developed here differ from the ordinary formulae of imbedding (that are also valid), only by their being written in terms of a covariant differentiation depending not only on $g_{(\beta\alpha)}$ but also on $g_{[\beta\alpha]}$. That means that they express geometrically besides the old well known facts only the situation of the imbedded space and its curves with respect to the bivector field $g_{[\beta\alpha]}$ and that all this could be expressed also without the use of a new covariant differentiation. *J. A. Schouten* (Epe).

Kurita, Minoru. On conformal Riemann spaces. J. Math. Soc. Japan **7**, 13–31 (1955).

After a summary of the standard theory of conformal correspondances between Riemann spaces, the author proves a series of theorems concerning conformally flat Riemann spaces of imbedding class one. Some of these are due to J. A. Schouten [Math. Z. **11**, 58–88 (1921)], M. Matsumoto [J. Math. Soc. Japan **3**, 306–309 (1951); MR **13**, 985], and Verbickiĭ [Trudy Sem. Vektor. Tenzor. Anal. **9**, 146–182 (1952); MR **14**, 795]. Of particular interest are two converses of a theorem of Schouten: If the principal curvatures of a hypersurface Σ in E^{n+1} $(n\geqq 3)$ are equal except one, then Σ is conformally flat. If a conformally flat Riemann space has its characteristic roots relative to a Euclidean space equal to p except for one, and if p satisfies a certain inequality, then the space is of imbedding class one.
C. B. Allendoerfer (Seattle, Wash.).

Roth, Millu. L'étude des directions enveloppantes dans un espace à connexion affine. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **3**, 123–262 (1952). (Romanian. Russian and French summaries)

If in an A_n $(n\geqq 3)$ with coordinates u^i at every point of a curve $u^i=u^i(\sigma)$ a vector θ^i is given and if the curve is mapped on the parallelly displaced tangent E_n, the images of the vectors θ^i envelope a curve if and only if the trivector $du^{[h}\theta^iD\theta^{j]}=0$, where D denotes the covariant differential. Instead of this trivector another trivector can be defined that transforms with a factor $(\sigma/\sigma^*)^3$ if the parameter σ is transformed. If θ^i satisfies this condition, its direction is called enveloping. If ξ^i is the vector in the direction of θ^i with the contact point as end point (principal vector), we get $D\xi^h=-du^h+\xi^hd\tau$, where $d\tau$ is an intrinsic parameter on the curve, called principal. By a change of coordinates this equation can be simplified. A special case arises if the enveloped curve reduces to a point. Then the directions θ^i are called concurrent because of their relation to concurrent vector fields [cf. Ricci-calculus, 2nd ed., Springer, Berlin, 1954, p. 322 ff.; MR **16**, 521]. If the tangent direction is concurrent, the curve is geodesic.

The second chapter deals with the V_n. For a V_n in V_{n+1} the curves for which the normals to V_n form a set of enveloping directions can be considered as generalised lines of curvature. It is proved that directions of a V_m in V_n that are enveloping for a curve in V_m have the same property for the curve looked upon as a curve of V_n. Many theorems are proved on enveloping and concurrent directions in a V_m or a set of V_m's in V_n. The third chapter deals with the metric properties of normalised and principal vectors in a V_m in V_n. In the last chapter normal systems of curves in A_2 in A_n are discussed. *J. A. Schouten* (Epe).

Atanasyan, V. A. Invariant rigging of surfaces of multidimensional affine spaces. Dokl. Akad. Nauk SSSR (N.S.) **98**, 701–704 (1954). (Russian)

We deal with a manifold V_n in a flat affine space A_m given by $\bar{r}=\bar{r}(x^1,x^2,\cdots,x^n)$. Each set of vectors $\bar{n}_1,\bar{n}_2,\cdots,\bar{n}_k$

which with $\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_m$ $(k=m-n)$, $\mathfrak{r}_i=\partial\mathfrak{r}/\partial x^i$, forms a set of m linear independent vectors forms a rigging. The vectors $\bar{n}_1, \bar{n}_2, \cdots, \bar{n}_k$ form a k-plane N_k. For each N_k the equations exist: $\mathfrak{r}_{ij}=\Gamma^h{}_{ij}\mathfrak{r}_h+h^\kappa{}_{ij}\bar{n}_\kappa$, $\bar{n}_{\kappa i}=\beta^h{}_{\kappa i}\mathfrak{r}_h+\Gamma^\lambda{}_{\kappa i}\bar{n}_\lambda$; $i, j, \cdots=1, \cdots, n$; $\kappa, \lambda=1, \cdots, k$. Then we have the Gauss-Codazzi-Ricci relations:

(A) $h^\kappa{}_{\underline{i}j}\beta^h{}_{\kappa\underline{k}}=R^h{}_{ijk}$ (Gauss); (B) $h^\kappa{}_{\underline{i}j|\underline{k}}=0$;

(C) $\beta^h{}_{\kappa\underline{i}|\underline{k}}=0$; (D) $h^\kappa{}_{\underline{m}j|}\beta^m{}_{\lambda\underline{k}}=R^\kappa{}_{\lambda jk}$,

where $R^h{}_{ijk}$ is the curvature tensor of the V_m and

$$R^\kappa{}_{\lambda jk}=\partial\Gamma^\kappa{}_{\lambda\underline{k}}/\partial x^{\underline{j}}+\Gamma^\sigma{}_{\lambda\underline{k}}\Gamma^\kappa{}_{\sigma\underline{j}};$$

the bar underneath the indices represents alternation. Introduced are the determinants T, F, E, B:

$$F=|F^{\kappa j}_{\lambda i}|=|n\delta_\lambda{}^\kappa\delta_i{}^j-h^\kappa{}_{im}h_\lambda{}^{mj}|,$$
$$E=|E^{\kappa j}_{\lambda i}|=|nk\delta_\lambda{}^\kappa\delta_i{}^j+2h^\sigma{}_{im}h_\lambda{}^{ml}h^\kappa{}_{ln}h_\sigma{}^{nj}|,$$
$$B=|B^{\kappa kl}_{\lambda ij}|=|n\delta_\lambda{}^\kappa\delta_i{}^k\delta_j{}^l+h^\kappa{}_{im}h_\lambda{}^{ml}\delta_j{}^k+h^\kappa{}_{jb}h_\lambda{}^{bl}\delta_i{}^k|,$$
(rows on κ, i; columns on λ, j)

and T is the determinant of the components of $h^\kappa{}_{ij}$ which, when $\neq 0$, allows us to pass from $h^\kappa{}_{ij}$ to $h_\kappa{}^{ij}$ such that

$$h^\kappa{}_{ik}h_\lambda{}^{ik}=n\delta_\lambda{}^\kappa, \quad h^\kappa{}_{ik}h_\kappa{}^{jk}=k\delta_i{}^j, \quad h^\kappa{}_{ik}h_\kappa{}^{ik}=nk.$$

Then it is shown that when $T\neq 0$, $E\neq 0$ the tensor

$$p_\lambda{}^m=h^\kappa{}_{ij|k}h_\lambda{}^{ij}h_\kappa{}^{km}$$

uniquely determines the rigging N_k; hence the rigging, for which $p_\lambda{}^m\neq 0$, is an invariant one. When $T, B, E, F\neq 0$ the $h^\kappa{}_{ij}$ and the V_n are called regular. Then (C) and (D) follow from (A) and (B), and the principal theorem of rigging can be given in the form: Given a regular tensor $h^\kappa{}_{ij}=h^\kappa{}_{ji}$ and a connection $\Gamma^h{}_{ij}=\Gamma^h{}_{ji}$ satisfying the equations (A), (B), and $p_\lambda{}^m=0$, then the V_n with intrinsic rigging N_k is determined but for an affine transformation of the A_n. Its $h^\kappa{}_{ij}$ and invariant N_k coincide with the given ones. Here

$$\beta^h{}_{\kappa k}=R^h{}_{ijl}h_\lambda{}^{ij}\bar{F}^{\lambda l}_{\kappa k},$$
$$\Gamma^\kappa{}_{\lambda l}=h_\sigma{}^{ij}\bar{F}^{\sigma r}_{\lambda k}(-\Gamma^m{}_{ij}h^\kappa{}_{mr}-\Gamma^n{}_{jr}h^\kappa{}_{in}+\partial h^\kappa{}_{ir}/\partial x^j),$$

where $\bar{F}^{\sigma r}_{\lambda k}$ are the reciprocals of the elements of the determinant F. *D. J. Struik* (Cambridge, Mass.).

Pisareva, N. M. On quadratic-fractional first integrals of geodesic curves of an affinely connected space. Mat. Sb. N.S. **36**(78), 169–200 (1955). (Russian)

The paper is concerned with spaces in which there exist quadratic-fractional first integrals of the equations of geodesics. The first part is concerned with 2-dimensional affinely connected spaces A_2 and Weyl spaces W_2. These results are then generalized to spaces A_n, W_n and V_n-Riemann spaces. The two theorems pertaining to A_2 are: In order that $a_{ij}du^idu^j/b_{ij}du^idu^j=C$ should be a first integral it is necessary and sufficient that any linear combination (with constant coefficients) of the two tensors should define a geodesic net. Equivalently it is necessary and sufficient that $a_{(ij,k)}=a_{(ij}N_{k)}$ and $b_{(ij,k)}=b_{(ij}N_{k)}$, where N_k is an arbitrary vector. By $W_2{}^\Gamma$ is understood a Weyl space for which the vector p_i is a gradient. It is then shown that an A_2 admitting a quadratic-fractional first integral is in geodesic correspondence with a $W_2{}^\Gamma$. A canonical form for the connection is then derived. The author considers next a space W_n with metric tensor g_{ij}, $g_{ij,k}=g_{ij}2\omega_k$ in which there exists another tensor a_{ij}, $a_{ij,k}=a_{ij}2\omega_k$. It is then shown that the characteristic roots for the principal directions of a_{ij} are necessarily constant and if ρ_1 is such a root of multiplicity m and $\underset{k}{x^i}$ are the corresponding vectors then $Dx^i=\underset{k}{x^i}a^k$, so that these vectors form a field of parallel m-flats and further, because these vectors form a holonomic system, the m-flats are tangent to a system of m-dimensional surfaces of the space. The last part deals with first integrals in A_n $(n>2)$ the approach and the necessary and sufficient conditions being somewhat different than for $n=2$. This is due to the fact that $a_{(ij,k)}=a_{(ij}N_{k)}$, $t_i=0$ and $a_{ij,k}=a_{ij}N_k$ are equivalent for A_2, but for A_n, $n>2$, the second implies the first, but not conversely. There are a number of bothersome misprints, particularly in the first part of the paper.

M. S. Knebelman (Pullman, Wash.).

Čech, Eduard. Géométrie projective différentielle des correspondances entre deux espaces. VIII. Čehoslovack. Mat. Ž. **4**(79), 143–174 (1954). (Russian. French summary)

This is the eighth of a series of papers dealing with projective correspondences [cf. MR **16**, 71]. In particular, this paper as well as the seventh deals with projective deformations of layers or strata of hypersurfaces in S_n. In paper VII the author distinguishes between general and special deformations. If $n\geqq 4$ the layer must be parabolic. If $n=3$ there exist special deformations for non-developable layers of hypersurfaces. The paper is devoted to the enumeration of the possible types of special deformations.

M. S. Knebelman (Pullman, Wash.).

Tachibana, Syun-ichi. On the imbedding of a projectively connected space in a projective space. Nat. Sci. Rep. Ochanomizu Univ. **5**, 5–9 (1954).

The author proves that an n-dimensional projectively connected space P_n can be locally imbedded in an (n^2-n+1)-dimensional projective space as an ordinary variety of bi-plane elements. Further, an n-dimensional projectively flat space can be imbedded in a $(2n-1)$-dimensional projective space as an ordinary variety of bi-plane elements. A bi-plane element is a configuration consisting of an n-plane and an $(N-n)$-plane having only one point in common; N is the dimension of the projective space. *C. B. Allendoerfer.*

Vasil'eva, M. V. Geometry of an integral. Mat. Sb. N.S. **36**(78), 57–92 (1955). (Russian)

Object of this study is the geometry of the integral $\iint F(x, y, z, p, q)\,dxdy$ invariant with respect to the infinite group of point transformations, following the ideas of E. Cartan, elaborated by V. V. Vagner [Trudy Sem. Vektor. Tenzor. Anal. **8**, 144–196 (1950); MR **13**, 777]. Geometrical objects of the first, second and third order are computed with an application to Riemannian geometry. Formulas are derived for the theory of curves, of surfaces and of congruences of curves. The paper opens with an exposition of the general ideas underlying this type of investigation, with special reference to G. F. Laptev's work on geometrical objects [Trudy Moskov. Mat. Obšč. **2**, 275–382 (1953); MR **15**, 254]. It ends with an outline of the case of an invariant m-uple integral in a manifold of n dimensions.

D. J. Struik (Cambridge, Mass.).

Berger, Erich R. Tensorflächen, Tensorellipsen und Tensorkreise. Österreich. Ing.-Arch. **8**, 231–236 (1954).

Sind λ_i die Eigenwerte eines symmetrischen Tensors so kann man vier "Tensorflächen" betrachten, die lineare, reziproke, quadratische und reziprok-quadratische; es sind sämtlich quadratische Flächen mit bzw. den Hauptachsenlängen $\lambda_i^{-1/2}$, $\lambda_i^{1/2}$, λ_i und λ_i^{-1}. In der Ebene fällt jede offen-

bar nach einer Drehung mit seiner reziproke zusammen. Es wird noch gezeigt wie man von einer kinematischen Erzeugung der quadratischen Tensorellipse aus zu den Kreisen von Culmann-Mohr und Mohr-Land kommt.

O. Bottema (Delft).

Case, K. M. Biquadratic spinor identities. Phys. Rev. (2) 97, 810–823 (1955).

The results of Brauer and Weyl [Amer. J. Math. 57, 425–449 (1935)] on the spin representations of orthogonal groups in n dimensions are used to expand the product of two spinors ψ_a and ϕ_b in terms of a linear basis for all matrices derived from the fundamental matrices $\Gamma(i)$ satisfying

$$\Gamma(i)\Gamma(j)+\Gamma(j)\Gamma(i)=2\delta_{ij} \quad (i, j=1, \cdots, n).$$

This result is then used to express the tensor formed in a particular way from two covariant and two contravariant spinors in terms of those formed in any other way. These results are used to obtain explicitly all biquadratic scalar and pseudo-scalar identities in case $n=2v$ and all scalar identities in $2v+1$ dimensions. The discussion is confined to the spin representations of the complex orthogonal group. The fact that the restriction to real groups implies the existence of an invariant anti-involution in the spin space is not discussed, nor is mention made of the existence of a decomposition of the direct product of a spin representation and its complex conjugate analogous to that given for the direct product of a spin representation with itself.

A. H. Taub (Urbana, Ill.).

Eriksson, H. A. S. Space reflection, time reversal and charge conjugation of spinor fields. Ark. Fys. 6, 349–358 (1953).

The author discusses the behavior of four-component spinor fields satisfying the Dirac equation under the spin images of the transformations corresponding to spatial reflections ($x^i=-x^i$, $t=t$, $i=1, 2, 3$) and time reversals ($x^i=x^i$, $t=-t$, $i=1, 2, 3$). His notation seems much more involved than is necessary in this problem and differs from that of earlier authors [e.g. R. Brauer and H. Weyl, Amer. J. Math. 57, 425–449 (1935); O. Veblen and J. von Neumann, Geometry of complex domains, Inst. Advanced Study, Princeton, 1936, rev. ed., 1955; MR 16, 516] who treated this problem in complete detail.

A. H. Taub.

Matschinski, Matthias. Sur les moyennes-tenseurs et sur leur application. C. R. Acad. Sci. Paris 239, 1457–1459 (1954).

The author defines a "tensor-mean" associated with a tensor of order n as a tensor of order $n+1$ but does not give a method for constructing the latter tensor. A prescription is given for writing differential equations for "tensor-means" from the differential equations satisfied by the tensors associated with the "tensor-means."

A. H. Taub.

Milkutat, Ernst. Explizite Darstellung des Kronecker-Tensors in einer Mannigfaltigkeit beliebig hoher Ordnung. Z. Naturf. 9a, 988 (1954).

Pan, T. K. A remark on the quotient law of tensors. Math. Mag. 28, 197–198 (1955).

NUMERICAL AND GRAPHICAL METHODS

***Vega, G. Tablicy seminznačnyh logarifmov. [Tables of seven-place logarithms.]** Izdat. Geodez. i Kartograf. Lit., Moscow, 1954. vi+pp. 5–560. 31 rubles.

Photocopy of the 65th stereotyped edition with the introduction translated into Russian.

***Karpov, K. A. Tablicy funkcii $w(z)=e^{-z^2}\int_0^z e^{x^2}dx$ v kompleksnoĭ oblasti. [Tables of the function $w(z)=e^{-z^2}\int_0^z e^{x^2}dx$ in a complex region.]** Izdat. Akad. Nauk SSSR, Moscow, 1954. 536 pp. (1 insert). 61 rubles.

The function $w(z)$ indicated in the title is tabulated to 5 decimals in the first octant of the complex plane of $z=\rho e^{i\theta}$ for $0<\theta<\pi/4$, $\rho\leqq 5$ and along the real axis for $\rho\leqq 10$ and the imaginary axis for $\rho\leqq 5$. The intervals $\Delta\rho$ and $\Delta\theta$ are chosen to accommodate the variations of $w(z)=u+iv$. The finest subdivisions in θ correspond to $\Delta\theta=\pi/288=.375°$ near $\theta=\pi/4$ and the coarsest to $\Delta\theta=\pi/36=5°$ for $0<\theta<20°$ and $0<\rho<3.5$; $\Delta\rho$ varies from .0002 to .01. This gives altogether a fairly comprehensive account of the function for the region considered. No differences are tabulated. Lagrangean interpolation is suggested in the introduction. Numerical coefficients for the asymptotic expansions of u and v are given as far as the first 6 terms. Graphs and reliefs of u and v are also shown. The table was apparently computed by hand and is intended primarily for use in radio wave propagation problems. This table is the first real step toward an adequate account of the error function in the complex plane.

D. H. Lehmer (Berkeley, Calif.).

***Karpov, K. A. Tablicy koėfficientov interpolyacionnoĭ formuly Lagranža. Priloženie k tablicam funkcii $w(z)=e^{-z^2}\int_0^z e^{x^2}dx$ v kompleksnoĭ oblasti. [Tables of coefficients of the Lagrange interpolation formula. Supplement to the tables of the function $w(z)=e^{-z^2}\int_0^z e^{x^2}dx$ in a complex region.]** Izdat. Akad. Nauk SSSR, Moscow, 1954. 79 pp.

These are tables of 4-point and 5-point Lagrangian interpolation coefficients at interval .001. Values of the coefficients are given to 6 decimals. The results were taken from the more extensive tables of the National Bureau of Standards [New York, 1944; these Rev. 5, 244]. The result is a convenient interpolation handbook for use with the other table mentioned in the title.

D. H. Lehmer.

Have

***A million random digits with 100,000 normal deviates.** By the RAND Corporation. The Free Press, Glencoe, Ill., 1955. xxv+200 pp. $10.00.

This work represents the largest published table of "random" digits and the largest published table of normal "deviates." The book is the collective effort of several members of the RAND Corporation. The introduction includes a brief discussion of how certain physical and numerical techniques were synthesized to produce the tables. Tabulated results of the various statistical tests that were applied to the tables are presented. The introduction concludes with an explanation of how these tables may be used. This book will no doubt be appreciated by research workers who have practical problems in such fields, for example, as

Experimental Design and Sampling. For sake of completeness and clarity it is hoped that future editions will contain a discussion as to what meaning should be attributed to the term "random" digit. *M. Muller* (Ithaca, N. Y.).

Berghuis, J. A table of some integrals. Math. Centrum, Amsterdam, Rekenafdeling. Rep. R 245, 8 pp. (1954).

Let

$$f_n(x)=\int_0^x v^n \tan v\,dv,\quad F_n(x)=\int_0^x v^n \tanh v\,dv,$$

$$g_n(x)=\int_0^x v^n \cot v\,dv,\quad G_n(x)=\int_0^x v^n \operatorname{cotanh} v\,dv.$$

The tables are f_n for $n=1(1)5$, $x=0(.05)1.5$; $g_n(x)$ for $n=1(1)5$, $x=0(.05)2.5$; F_n and G_n both for $n=1(1)4$, $x=0(.02)1.98$. They are given to 7 or 8 D with an error stated to be at most 1×10^{-7} in the case of g, F, G and at most 3×10^{-8} in f. The tables have been checked by differencing; f and g were calculated by hand; F and G were computed on the relay machine ARRA of the Math. Centrum. *John Todd* (Washington, D. C.).

Ishiguro, Eiichi, Yuasa, Sayoko, Sakamoto, Michiko, and Arai, Tadashi. Tables useful for the calculation of the molecular integrals. VI. Nat. Sci. Rep. Ochanomizu Univ. 5, 33–58 (1954).

[For part V see Ishiguro, Yuasa, Sakamoto, and Kimura, same Rep. 4, 176–191 (1954); MR 16, 175.] Table XXII gives values to 11 or more significant figures of

$$\left(\frac{2}{\pi}\right)^{1/2} K_{n+1/2}(\alpha)\quad\text{and}\quad (2\pi)^{1/2}I_{n+1/2}(\alpha)$$

for 18 selected values of α ranging from 3 to 35.5, and for each α for $n=-2(1)5$ except that in 3 cases $n=5$, and in 3 other cases $n=4$, 5 are omitted. Table XXIII gives 18 pp. of selected values of

$$\mathbf{P}_{n,\,l+1/2}(\kappa,\tau)=\mathbf{Z}_{1,\,n,\,l+1/2}(\kappa,\tau),$$
$$\mathbf{Q}_{n,\,l+1/2}(\kappa,\tau)=\mathbf{Z}_{2,\,n,\,l+1/2}(\kappa,\tau),$$

where the $\mathbf{Z}$ are the functions defined by Barnett and Coulson [Philos. Trans. Roy. Soc. London. Ser. A. 243, 221–249 (1951); MR 12, 702]. *A. Erdélyi* (Pasadena, Calif.).

Sternberg, R. L., and Shipman, J. S., and Thurston, W. B. Tables of Bennett functions for the two-frequency modulation product problem for the half-wave linear rectifier. Quart. J. Mech. Appl. Math. 7, 505–511 (1954).

Tables of

$$A_{mn}(k)=2\pi^{-2}\iint(\cos u+k\cos v)\cos mu\cos nv\,du dv,$$

where the integral is over that part of the square $0\leqq u\leqq\pi$, $0\leqq v\leqq\pi$, for which $\cos u+k\cos v\geqq0$, are given to 8D for $(m\,n)=(0\,0)$, $(0\,1)$, $(2\,0)$, $(1\,1)$, $(0\,2)$, $(4\,0)$, $(3\,1)$, $(2\,2)$, $(1\,3)$, $(0\,4)$, and for $k=0.02(.02)1$. Actually $\frac{1}{2}A_{00}$ is tabulated, not A_{00}; also $A_{mn}\equiv0$ if $m+n$ is an odd integer greater than 1.

For $0<k\leqq.40$, five to nine terms of the ascending power series in k were used. For the range $.42\leqq k\leqq.98$ use was made of relations of the form

$$A_{mn}(k)=k^{-n}[P_{mn}(k)E(k)+Q_{mn}(k)K(k)],$$

where the P_{mn}, Q_{mn} are polynomials in k, and of the tables of elliptic integrals computed by A. Fletcher [Phil. Mag. (7) 30, 516–519 (1940); MR 2, 239]. Expressions in closed form are available for $k=0$, $k=1$.

Recurrence relations are given which enable the functions with larger m, n to be obtained; it is noted that accuracy is lost in the use of some of them for small k. A thorough discussion of the details of calculations, and of the careful checking carried out is given; there are also remarks on interpolation. *John Todd* (New York, N. Y.).

***Rašković, Danilo P. Osnovi numeričkog računanja. [The elements of numerical computation.]** Građevinska Knjiga, Belgrade, 1954. viii+168 pp.

This is an elementary and abbreviated account of numerical analysis from a hand-computing point of view. Topics considered are: Arithmetic of real and complex numbers, approximate computation, rational operations with approximate numbers, error estimation, interpolation, numerical solution of equations and numerical integration. The book is replete with examples but short on theory. For example in the treatment of the Graeffe process there is no discussion of the case of complex roots. *D. H. Lehmer.*

***Peltier, J. Calcul de certaines fonctions usuelles en système binaire.** Les machines à calculer et la pensée humaine, pp. 295–305. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

The author suggests that (a) because the operations of rational arithmetic are binary and (b) many large-scale computers are binary computers that there is more than just a play on words involved. It is the purpose of his paper to show how the connection between (a) and (b) can be exploited to design computation schemes for $\log x$, $x^{1/n}$ and a^x. Frequent use is made of the expansion

$$\log x=2\sum_{n=1}^{\infty}t^{2n-1}(2n-1)^{-1},\quad t=(x-1)/(x+1).$$

The following simple example gives an idea of the methods used: To compute $2^{1/7}$ we first compute 2^α, where

$$\alpha=(7\cdot2^9)^{-1}=1/3584.$$

This is simply $2^\alpha=(1+t)/(1-t)$, where $t=\frac{1}{2}\alpha\log 2$. Having found 2^α, the desired $2^{1/7}$ is the result of nine successive squarings. *D. H. Lehmer* (Berkeley, Calif.).

***Smith, E. J. Logical approach to the design of computer circuits.** Proceedings of the symposium on information networks, New York, April, 1954, pp. 249–266. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1955.

The author surveys two methods of 'minimizing' representative Boolean functions, namely the Harvard minimizing chart [Synthesis of electronic computing and control circuits, Harvard Univ. Press, 1951; MR 13, 497] and the Karnaugh map method [Trans. Amer. Inst. Elec. Engrs. Part I. 72, 593–599 (1953)] and discusses the synthesis of 'computer-type' circuits from representative Boolean functions. *S. Gorn* (Aberdeen, Md.).

Hermes, Hans. Die Universalität programmgesteuerter Rechenmaschinen. Math.-Phys. Semesterber. 4, 42–53 (1954).

The author is concerned with problems which are related to those considered by Turing in his thesis. He shows that every function which is computable can be computed by a sequence-controlled calculator. He gives a number of examples. *H. H. Goldstine* (Princeton, N. J.).

Gorn, Saul. The automatic analysis and control of computing errors. J. Soc. Indust. Appl. Math. **2**, 69–81 (1954).

The author is interested in seeing how far it is possible to go in the direction of automatic analysis of errors due to the finite character of calculating machines. A number of examples are considered in some detail. These include among others linear algebraic equations and ordinary differential equations. *H. H. Goldstine.*

★van den Dungen, F. H. Sur le contrôle des intégrations numériques. Studies in mathematics and mechanics presented to Richard von Mises, pp. 103–110. Academic Press Inc., New York, 1954. $9.00.

The author discusses a technique for controlling the propagation of errors in numerical solutions of linear hyperbolic equations. Use is made of certain "variant" integrals to derive smoother finite-difference schemes than are ordinarily used for the wave equation. *E. Isaacson.*

Davis, Philip, and Rabinowitz, Philip. A multiple purpose orthonormalizing code and its uses. J. Assoc. Comput. Mach. **1**, 183–191 (1954).

It is frequently quite useful to replace a sequence of functions by an orthonormal one by means of the Gram-Schmidt process. The authors have written a code for use on the National Bureau of Standards Eastern Automatic Computer for this purpose. The details of the work are discussed in the paper; in particular, there is a discussion of the handling of those numbers which are critical in size. The usefulness of the code is indicated by the authors for various purposes, eight of which are enumerated and discussed.

H. H. Goldstine (Princeton, N. J.).

★Householder, A. S. Errors in iterative solutions of linear systems. Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 30–33. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

Let x_0 be an approximate solution of the linear algebraic system $Ax=y$. Let $r_0=y-Ax_0$. Let $r_0^*=y-(Ax_0)^*$, where the stars denote a "digital" realization of the former formula (i.e., one including all rounding-off errors of a certain digital-computer routine). Let C be an approximate inverse of A. Let $B=I-AC$; $B^*=I-(AC)^*$. When B is small, x_0+Cr_0 is known to be closer to $A^{-1}y$ than x_0 is. Form the digital approximation $x_1=x_0+(Cr_0^*)^*$. The paper considers the question: if $y_1=y-Ax_1$, is $N(r_1)<N(r_0)$ for a given norm function N?

A sufficient condition for the answer "yes" is that

$$(*)\quad N(r_0^*)>\{2N(r_0-r_0^*)+N(A)N[Cr_0^*-(Cr_0^*)^*]D^{-1},$$

where $D=1-N(B^*)-N(B-B^*)$. Sufficient conditions for (*) are given for four different methods of rounding off the matrix products Ax_0 and $(Cr_0^*)^*$, and various associated methods of scaling. The Seidel iterative process is briefly mentioned. [Note: The unusual (among professional mathematicians) division symbol ÷ can easily be misread as a + sign in this printing.] *G. E. Forsythe.*

★Lanczos, Cornelius. Chebyshev polynomials in the solution of large-scale linear systems. Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 124–133. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

To solve a linear algebraic system $Ay=b$, the author first forms the positive definite system $C_0y=c_0$. Here $C_0=A^*A/\mu$, $c_0=A^*b/\mu$, and the number μ is so chosen that the eigenvalues λ_i of C_0 lie in the interval $(0, 1)$. He then considers approximations to y of form (*) $y_m=G_m(C_0)c_0$, where G_m is a polynomial of degree m. Since

$$r_m=c_0-C_0y_m=\{I-C_0G_m(C_0)\}c_0,$$

the object is to find

$$F_{m+1}(\lambda)=1-\lambda G_m(\lambda)$$

such that $F_{m+1}(0)=1$, while all $F_{m+1}(\lambda_i)$ are as small as possible. Since the λ_i are known only to lie in the interval $(0, 1)$, one searches for an $F_{m+1}(\lambda)$ which is as nearly zero as possible for $0<\lambda<1$.

After the change of variables $\lambda=\sin^2(\theta/2)$, the author proposes the Fejér kernel in θ as a suitable polynomial. Returning to λ, he develops recursion formulas for $G_m(\lambda)$ in terms of the Chebyshev polynomials $T_k(\lambda)=\cos k\theta$. Other changes of variable yield polynomials with simpler coefficients. The Dirichlet kernel is used similarly. Estimates are given for the smallness of $F_{m+1}(\lambda)$ for λ near 0. With the Fejér kernel, for example, if $m=6$ and min $\lambda_i \doteq .025$, then five applications of (*) make all $|F_{m+1}(\lambda_i)|^5 \leqq .05$. The order of C_0 is irrelevant; only the "spread" $P=\max\lambda_i/\min\lambda_i$ [called by other authors the "condition" of C_0] is relevant to the success of the method.

The author's method with $m \doteq 6$ is considered applicable for systems with only moderate spread or "skewness"—perhaps $P<100$.

To permit applications of the method to systems with P up to perhaps 1000, the author discusses first decreasing the spread of C_0 [i.e., "preconditioning" C_0] by premultiplying C_0 by a polynomial $Q_m(C_0)$. Again Chebyshev polynomials prove useful, because of their known range and steep slopes near the ends of their interval.

The author lists four applications of the method reported here: (1) Solving linear systems of moderate skewness. (2) Modifying an orthogonal function system to preserve orthogonality after somewhat altering the domain. (3) Eigenvalue analysis [the author suggests getting complex eigenvalues by the trick reported independently by Azbelev and Vinograd, Dokl. Akad. Nauk SSSR (N.S.) **83**, 173–174 (1952); MR **14**, 126]. (4) Combination with the author's method of minimized iterations [J. Res. Nat. Bur. Standards **49**, 33–53 (1952); MR **14**, 501] to deal with systems with $P \gg 1000$.

There are some remarks on stability of linear systems, and on the physical implausibility of systems with large P.

G. E. Forsythe (Los Angeles, Calif.).

Horvay, G. Solution of large equation systems and eigenvalue problems by Lanczos' matrix iteration method. General Electric Company, Knolls Atomic Power Laboratory, Schenectady, N. Y., Rep. no. **KAPL-1004**, 113 pp. (1953). $.70 (may be obtained from Office of Technical Services, Department of Commerce, Washington 25, D. C.).

This is an exposition, with numerical examples from network theory, of the following old and new material in linear algebra: algebra of matrices; eigenvalues of hermitian matrices; classical iteration method for computing eigenvalues; method of minimized iterations [C. Lanczos, J. Res. Nat. Bur. Standards **45**, 255–282 (1950); MR **13**, 163]; inversion of large matrices by minimized iterations [Lanczos, ibid. **49**, 33–53 (1952); MR **14**, 501]; inverting matrices by use of Chebyshev polynomials [see the paper reviewed above]; inverting matrices which are almost blockwise diagonal

[sources include G. Kron, J. Appl. Phys. 24, 965–980 (1953); MR 15, 747; and G. Horvay, J. Appl. Mech. 19, 355–360 (1952)]. There are 26 references, 25 figures, and 7 tables.

The article grew out of lecture courses presented [to engineers?] in the General Electric Co. and various universities. The author remarks on the great didactical value to engineers of Lanczos's methods in stressing the value of vector spaces, Fourier expansions, and other material from classical mathematical physics. *G. E. Forsythe.*

Lemke, C. E. The dual method of solving the linear programming problem. Naval Res. Logist. Quart. 1, 36–47 (1954).

The author is concerned with the problem of minimizing $\sum a_j x_j$ subject to $\sum b_{ij} x_j = c_i$. This is, as is well known, equivalent to the dual problem of maximizing $\sum c_i u_i$ subject to $\sum b_{ij} u_i = a_j$. He gives the details of an iterative process for computing the solution to the dual problem for which the following advantages over previous methods are claimed: (i) should it not be required that the variables be non-negative, the number of variables need not be doubled, (ii) it considers simultaneously the direct and dual problem, and (iii) it uses the original data and the current inverse matrix (rather than iterated data and the current matrix) at each stage, thus eliminating one source of accumulated errors. *J. M. Danskin* (Washington, D. C.).

Routledge, N. A., Lord, W. T., and Eminton, E. Note on the evaluation of the integral

$$I = -\int_0^1\int_0^1 S''(x)S''(\xi)\log|x-\xi|\,dxd\xi,$$

for $S'(0)=0=S'(1)$. J. Roy. Aero. Soc. 58, 787–788 (1954).

The integral I, with the function $S(x)$ given graphically, is evaluated by representing $S(x)$ in the form

$$S(x) = S_0(x) + T(x)$$

with

$$S_0(x) = S(0) + \frac{1}{\pi}[S(1)-S(0)][\theta - \tfrac{1}{2}\sin 2\theta],$$

where $x=\frac{1}{2}(1-\cos\theta)$ $(0\leq x\leq 1)$. ($S_0(x)$ has some aerodynamic significance.) $T(x)$ is represented by a Fourier sine series determined by the standard techniques of numerical harmonic analysis. The integral I can finally be calculated in terms of the Fourier coefficients of $S'(0)$ which are readily expressible in terms of the coefficients of $T(x)$.

E. Isaacson (New York, N. Y.).

Todd, John. Experiments in the solution of differential equations by Monte Carlo methods. J. Washington Acad. Sci. 44, 377–381 (1954).

The paper gives the results of certain numerical experiments carried out on the National Bureau of Standards Eastern Automatic Computer (SEAC). The experiments were aimed at trying out Monte Carlo methods for estimating statistically the solution of the differential equation $\Delta V=0$ in the unit square in the (x, y)-plane, with boundary values all equal to zero except on the line $y=1$, where $V(x, 1)=\sin \pi x$. The differential equation is replaced by the simplest analogous difference equation, using a square lattice. The solution is then estimated by repeated realizations of a random-walk procedure first suggested by Courant, Friedrichs, and Lewy [Math. Ann. 100, 32–74 (1928)] in which a particle starting at the point (x, y) on the lattice is supposed to move with probability 1/4 to each of its neighbors until it reaches a boundary point, whereupon a score is tallied which equals the boundary value at that boundary point. Tables are presented which give the arithmetic means and cumulative means of successive groups of 64 numerical realizations of this stochastic process, corresponding to four different starting points in the unit square. The pseudo-random numbers used were the numbers $a_n=2^{-42}b_n$, $b_{n+1}=bb_n \pmod{2^{42}}$, $b_0=1$, $b=5^{17}$. An experiment in which importance sampling is tried out is also discussed. With the simplest procedure, each run of 64 random walks took 20 seconds on the SEAC, and 5 minutes with the importance sampling procedure. Some analogous numerical results are given for the three-dimensional case. *J. H. Curtiss.*

Breiter, Mark C. Study of parameters in a differential equation related to blast. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. No. 822, 13 pp. (1954).

The Runge-Kutta method is used to integrate a linear second-order ordinary differential equation with discontinuous coefficients. The aim is to investigate the dependence of the solution on two critical constants which appear in the differential equation. A completely automatic program for carrying out the numerical analysis is described.

E. Isaacson (New York, N. Y.).

Budden, K. G. The numerical solution of differential equations governing reflexion of long radio waves from the ionosphere. Proc. Roy. Soc. London. Ser. A. 227, 516–537 (1955).

"Two methods are described for obtaining numerical solutions of the differential equations which govern the reflexion of long and very long radio waves from the ionosphere at vertical or oblique incidence.

"In the first method the first-order simultaneous equations, derived from Maxwell's equations and the constitutive relations for the ionosphere, are integrated by a step-by-step process proceeding downwards. The integrations are started from properly chosen initial solutions. From the resulting field variables at the bottom of the ionosphere a reflexion coefficient matrix R is derived, whose elements include the familiar reflexion coefficients. Two integrations are needed for each derivation of the elements of R.

"For the second method, it is shown that the formulae for R for a level below the ionosphere can be applied also within the ionized medium, and define a more general matrix variable whose elements are the dependent variables in a new set of differential equations. These are integrated by a step-by-step process as in the first method. The solution obtained below the ionosphere gives the required set of reflexion coefficients without further calculation. Only one integration is required for each derivation.

"The equations are given in full for certain important special cases."

"The first method of calculation has been used for a long series of calculations, mainly for waves of frequency 16 kc/s. The second method, however, gives the results more directly. It has been used to repeat some of the calculations made by the first method, and the results from the two methods agree, thus providing a check of both methods. The second method is now being used for some calculations at 70 kc/s, and may later be applied to even higher frequencies." *E. Isaacson* (New York, N. Y.).

Spirin, G. M. Improvement of the iterative method of solution of the biharmonic finite-difference equation. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 292–295 (1954). (Ukrainian. Russian summary)

Sokołowski, Marek. Application of the perturbation method to plate problems. Arch. Mech. Stos. **5**, 415–436 (1953). (Polish. Russian and English summaries)

The perturbation method was introduced by H. Poincaré [Leçons de mécanique céleste, Gauthier-Villars, Paris, 1905] to find approximate solutions of differential equations. It permits one to solve eigenvalue and boundary-value problems differing slightly from well known and already solved cases. As a special case of the perturbation method, a one-parameter eigenvalue problem was presented in detail by L. Collatz [Eigenwertaufgaben mit technischen Anwendungen, Akademische Verlagsgesellschaft, Leipzig, 1949; MR **11**, 137]. The theory given by the author is based on Collatz's monograph. The author applies the method to stability problems of isotropic and orthotropic rectangular plates compressed non-uniformly along opposite edges. He compares his results with exact solutions and shows that the error is negligible. The method simplifies considerably in case of simple boundary-value problems without eigenvalues. *T. Leser* (Aberdeen, Md.).

Klovskiĭ, D. D. Approximate graphical-analytic method of construction of the frequency characteristics of a system from its transfer characteristic. Ž. Tehn. Fiz. **25**, 333–338 (1955). (Russian)

Radoslovich, E. W., and Megaw, Helen D. Calculation of geometrical structure factors for space groups of low symmetry. I. Acta Cryst. **8**, 95–98 (1955).

This paper describes a simple calculator for the function $\cos(hx+ky+lz)$. Values of this function can be read directly from tables of $\cos hx$, provided that the origin of the latter can be shifted an amount $(ky+lz)$ at will. A simple mechanical device to do this is described. *Author's summary.*

Laško, O. S. A new trigonometric method of computing the distribution curve of the atoms of a liquid from X-ray data. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 150–157 (1953). (Ukrainian. Russian summary)

Bennett, Albert A. Some numerical computations in ordnance problems. Comm. Pure Appl. Math. **8**, 117–122 (1955).

Scanlan, Robert. Solution de quelques cas de stabilité de structures au moyen de réseaux résistifs superposés. C. R. Acad. Sci. Paris **240**, 1047–1050 (1955).

Redshaw, S. C. The determination of the pressure distribution over an aerofoil surface by means of an electrical potential analyser. Aeronaut. Res. Council, Rep. and Memo. no. **2915** (1952), 40 pp. (1954).

Gładysz, S., and Rybarski, A. On modelling three-dimensional fields by a plane field of current. Zastos. Mat. **2**, 150–160 (1955). (Polish. Russian and English summaries)

A three-dimensional electrostatic field with the potential ψ, expressed in an orthogonal curvilinear system (u, v, w), does not depend on one variable. The problem consists in finding the necessary and sufficient conditions of modelling the field ψ by an electric field in the plane. Modelling cannot depend on the function ψ. The required conditions have been obtained and it has been found that they are satisfied only for a certain class of curvilinear systems. For that class a method of modelling boundary conditions is also given. The results are illustrated by the example of cylindrical coordinates. *Author's summary.*

Bondar, M. G. Electrical modelling of vibration and stability of bar systems. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 375–382 (1953). (Ukrainian. Russian summary)

D'yačenko, V. E., and Tancyura, M. A. On electromodelling of equations of elliptic type. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 143–149 (1953). (Ukrainian. Russian summary)

Šura-Bura, M. R. On the solution on electric networks of a finite-difference equation approximating the Dirichlet problem for the Laplace equation. Vyčisl. Mat. Vyčisl. Tehn. **1**, 46–55 (1953). (Russian)

Detailed exposition of work reported in Dokl. Akad. Nauk SSSR (N.S.) **78**, 21–24 (1951); MR **13**, 693.

Rowe, H. T. The calculator NORC of the International Business Machines Corp. Calc. Automat. Cibernet. **4**, no. 9, 43–53 (1955). (Spanish)

Bonfiglioli, Luisa. Determination of limits by nomographic charts. Riveon Lematematika **8**, 33–40 (1954). (Hebrew. English summary)

The isosceles nomograph, consisting of two scales perpendicular and the third inclined at 45° to them, is employed to describe a function of the type $y=f(x)/\varphi(x)$. From the nomograph the following are found: (1) $\lim_{x\to x_0} f(x)/\varphi(x)$, where $\lim_{x\to x_0} f(x)=0$, $\lim_{x\to x_0} \varphi(x)=0$; (2) maximum and minimum values of y; (3) maximum and minimum values of y when a side condition on f and φ is given. *Hirsch Cohen* (Pittsburgh, Pa.).

Gärtner, G. Die zweckmässige Darstellung von Gleichungen mit vier bis sechs Veränderlichen in Nomogrammen. V. D. I. Z. **97**, 13–15 (1955).

Zalts, K. Ya. On nomographing the function

$$F_1K_{23}+F_2L_{31}+F_3M_{12}$$

without quadratures. Mat. Sb. N.S. **33(75)**, 383–388 (1953). (Russian)

The vanishing of four third-order determinants involving third and lower order derivatives of K, L and M, and the non-vanishing of a second-order determinant, is shown to be necessary (and said to be sufficient) for the representation

$$F_1K+F_2L+F_3M=\begin{vmatrix} F_1 & G_1 & H_1 \\ F_2 & G_2 & H_2 \\ F_3 & G_3 & H_3 \end{vmatrix},$$

where subscripts indicate dependence on corresponding variables and double subscripts have been dropped. Using, for example, $[K_2']$ to denote that the variable z_2 in $\partial K/\partial z_2$ has been replaced by a constant ζ_2, a solution is $G_1=[L_2']$, $H_1=[M_3']$, $G_2=-[K_3']$, $H_3=-[K_2']$, $G_3=Z_3$, $H_2=Y_2$, where the constants ζ_2 and ζ_3 and the functions Y_2 and Z_3

are taken to satisfy

$$[K_2'][K_3']-Y_2Z_3=K,$$
$$[K_2'][L_3']+[M_2']Z_3=L,$$
$$[M_2'][K_3']+Y_3[L_3']=M.$$

Effective means for satisfying these conditions are not discussed. *R. Church* (Monterey, Calif.).

Vil'ner, I. A. The problem of anamorphosis. Dokl. Akad. Nauk SSSR (N.S.) **77**, 177–180 (1951). (Russian)

The main steps leading to the results summarized in the first paragraph of the review of the author's earlier paper [same Dokl. (N.S.) **58**, 729–732 (1947); MR **9**, 534] are here arranged in more logical form with complete statement of the hypotheses of the three theorems. Equations (2) of the review referred to were quoted from the top of page 730; the present work shows that in each of these equations the sign preceding the coefficient 3 should be plus. A further result is given: the x and y scales lie on a conic if $\Delta_{1y}=-\Delta_{2x}\neq 0$ and $\Delta_{1xy}=-\Delta_1\Delta_{1y}$. On pages 238 to 240 of the paper reviewed below, the present paper appears with practically no change, while a somewhat different arrangement of the material containing detailed arguments and discussions occupies pages 133 to 148. *R. Church.*

***Vil'ner, I. A. Nomographing systems of equations and analytic functions.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 125–242. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

The five chapters of this paper, to a large extent independent of each other, and yet profusely cross-referenced to each other and the author's previous work, provide a self-contained and rather complete exposition of the area treated. The first four sections briefly review the work of others concerned with the representation of $f(x, y, z)=0$ as a Massau determinant. Some results related to the two papers reviewed together below are here stated and in the fifth section (pp. 149–152) the conditions given in those papers are applied to $F=z_3-u(z_1, z_2)$. This section, mainly devoted to nomographing a system $f_1(x, y, z_1)=0$ and $f_2(x, y, z_2)=0$, presents proofs, discussion and applications of the material published in the paper reviewed immediately above which constitutes the final section of the present paper. In the sixth section attention is turned to a pair of conjugate harmonic functions, and much of the rest of the paper, especially Chapter III, is here outlined. Further topics treated in the first chapter include: nomographic consideration of an analytic function as the result of eliminating auxiliary variables (17 examples); an outline of the author's method of binary fields for n equations in $n+2$ variables; applications to space nomograms (using descriptive geometry) and to nomograms with a transparent element; relation of nomography to the geometry of nets.

The second chapter, on systems of equations of the first nomographic classes, reproduces an earlier paper [Prikl. Mat. Meh. (N.S.) **4**, no. 2, 105–116 (1940); MR **9**, 534] augmented near the beginning with four pages in which alternative formulations of the necessary and sufficient conditions of that paper are proved. Chapter III is devoted to finding all analytic functions $F(z, w)=0$ of the first nomographic class. The conditions for this lead to consideration of five cases from which seven elementary and two non-elementary (elliptic integral) normal forms result. In Chapter IV the corresponding canonical representations are derived in explicit form free of quadratures and finally, for the non-elementary normal forms, they are presented very nearly as in two earlier papers [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **53**, 187–190 (1946); **55**, 783–786 (1947); MR **8**, 494; **9**, 106] in which the canonical representations for the non-elementary normal forms here given were not included. This chapter concludes with a brief summary of results on nomographing analytic functions in polar coordinates and on functions of the second nomographic class, both treated in more detail elsewhere [Mat. Sb. N.S. **27**(69), 3–46 (1950); MR **14**, 1021]. In the paper just referred to the author mentions the nomographic representation of the equations of a plane projective transformation. Details of this occupy nine pages of the concluding chapter in which are discussed other applications, including nomographic representation of nine coordinate transformations from the domain of cartography. *R. Church* (Monterey, Calif.).

***Vil'ner, I. A. On a nomographic problem.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 253–259. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Using quite elementary methods, the author obtains Massau determinants equivalent to the following symmetric equations of third nomographic order:

$$A_0+A_1\sum f_1+A_2\sum f_1f_2+A_0f_1f_2f_3=0,$$

$$\sum f_1f_2+\sum\frac{1}{f_1f_2}=0.$$

It is pointed out that the solution of these problems is contained in the general investigations of O. V. Ermolova [Uč. Zap. Moskov. Gos. Univ. Nomografiya **28**, 55–70 (1939); MR **1**, 256] and A. I. Moldaver [ibid. **28**, 75–106 (1939); MR **1**, 256] and was obtained by the author in the form here given, but not published, in 1932 and 1933. The second equation is a canonical form for an equation representable by a nomograph with scales on a fourth-order curve with one real and two imaginary double points. See reference to this paper in the following review. *R. Church.*

Vil'ner, I. A. Algebraic solution of the problem of anamorphosis of functions in invariant form. Dokl. Akad. Nauk SSSR (N.S.) **90**, 5–8 (1953). (Russian)

Vil'ner, I. A. Solution of the problem of anamorphosis of functions in $(N-1)$-dimensional space by vector-algebraic methods. Uspehi Mat. Nauk (N.S.) **8**, no. 3(55), 153–156 (1953). (Russian)

In the first of these two papers it is stated that the vanishing of the fourth order determinant $|a_{ij}|$, $a_{ij}=F_{z_3^{j-1}z_1^{i-1}z_2^{i-1}}$, is necessary and sufficient for the representation as a scalar product:

$$F(z_1, z_2, z_3)=\sum_{i=1}^{3}a_i(z_3)b_i(z_1, z_2).$$

Further necessary and sufficient conditions for the representation of F as a scalar triple product (Massau determinant) $\bar{a}_3\cdot\bar{a}_1\times\bar{a}_2$, where each vector $\bar{a}_i$ depends only on the corresponding variable z_i, are the vanishing of two more determinants of third order: $|a_{ij}|$, $a_{ij}=F_{z_3^j z_k^{i-1}}$, $k=1, 2$. If it is necessary or desirable to do so, the symbols such as $F_{z_j^2}$ can be taken to mean $F(z_1^2, z_2, z_3)$ instead of derivatives, where $z_i^0=z_i$ and the z_i^1, z_i^2, etc., $i=1, 2, 3, \cdots$ are new variables. The use and properties of these "symbolic derivatives" and similar "differentials" is a significant feature of this work.

The second paper is a report of a lecture. Here N variables z_i are considered, $2N-1$ determinants of order N and one of order $N+1$ are defined, and in terms of N of them results analogous to the above are given. In addition, explicit for-

mulas for the Nth order Massau determinant (the elements are certain adjoints of N of the determinants mentioned) and the anamorphizing factor of the form $\varphi(z_N)\psi(z_1, \cdots, z_{N-1})$ are stated. Another formulation of these conditions in terms of N determinants of order $N+1$ and $N-1$ determinants of order N is given and stated (for $N=3$) at the end of the paper reviewed immediately above where application is made assuming F satisfies certain restrictions involving symmetry. Reference is made to conditions for simultaneously nomographing a system

$$f_i(z_1, \cdots, z_{N-1}; z_{N+i-1})=0 \quad (i=1, \cdots, N-1)$$

in suitable hyperspaces and to the nature of such a representation. *R. Church* (Monterey, Calif.).

Vil'ner, I. A. The problem of anamorphosis for analytic functions of a complex variable and N-functional equations. Dokl. Akad. Nauk SSSR (N.S.) **83**, 341–344 (1952). (Russian)

This paper presents formulations of criteria for nomographability, class and genus of an analytic function $\Phi(w, z)=0$, where $w=u+iv$ and $z=a+ib$, which are more concise than those previously given [same Dokl. (N.S.) **63**, 99–102 (1948); MR **10**, 577]. This is done in terms of a parameter briefly introduced before [Mat. Sb. N.S. **27**(69), 3–46 (1950); MR **14**, 1021], when condition (5) below was stated: $\eta=\ln \varphi\bar{\varphi}(\varphi^2-\bar{\varphi}^2)^{-1/2}$, where $\varphi(z)=dw/dz$ and the bar denotes conjugate. η is related to the differential parameters for an arbitrary system of two equations [Prikl. Mat. Meh. (N.S.) **4**, no. 2, 105–116 (1940); MR **9**, 534] by $2\eta_a=\Delta_1$ and $-2\eta_b=\Delta_2$. Necessary and sufficient conditions for nomographability are: (1) $\eta_{aab}=2\eta_a\eta_{bb}$ and $\eta_{bba}=-2\eta_b\eta_{ab}$ or (2) $(K_4)_z=0$ and $(K_4)_{\bar{z}}=0$, where $K_4=e^{2\eta}(\eta_{zz}-\eta_{\bar{z}\bar{z}})$ or (3) that K_4 be constant. A necessary and sufficient condition for $\Phi(w, z)=0$ to be of the first class with z-scales straight is (4) $\eta_{ab}=0$ or (5) $\eta_{zz}-\eta_{\bar{z}\bar{z}}=0$. This problem of anamorphosis can be considered as a system of two functional equations in a and b to which its solution then applies.

R. Church (Monterey, Calif.).

Vil'ner, I. A. Nomograms for the computation of elliptic functions and integrals. Uspehi Mat. Nauk (N.S.) **9**, no. 2(60), 113–124 (1954). (Russian)

The first nomogram was described in some detail when an earlier paper [Uspehi Mat. Nauk (N.S.) **2**, no. 6(22), 227–237 (1947); MR **10**, 577] in which it was published in essentially the present form was reviewed. The second consists of orthogonal rectilinear scales for a and b, a family of concentric circles for p and q and a family of equilateral hyperbolas passing through two fixed points for m. With one alignment it permits solution of

$$p+iq=M\int_0^s \frac{d\xi}{(1-k^2\sin^2\xi)^{1/2}},$$

$s=a+ib$, when $|k'|=1$. Here $k^2+k'^2=1$, $m=(1+k')^2/4k'$ and $M=k/|k|i^{1/2}$. Five auxiliary graphs are given for determination of m, k' and other related quantities. The third nomogram, less detailed than the others, is constructed for $M=k/(1+i)$. It differs from the second only in the form of the curves for p and q (not conics). The fourth nomogram is a projective transformation of the second in which the scale for b is at infinity, the curves for m are circles with a common chord (the scale for a) and the curves for p and q are confocal hyperbolas.

The author showed the existence of such nomograms earlier and gave [C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 783–786 (1947); MR **9**, 106] the canonical representation and defining equations. Applications to the solution of certain equations involving elliptic integrals are outlined. Approximate formulas for elliptic functions that have been obtained from these investigations are given.

R. Church (Monterey, Calif.).

RELATIVITY

Tonnelat, Marie-Antoinette. La solution générale des équations d'Einstein $g_{\underset{+}{\mu}\underset{-}{\nu};\rho}=0$. J. Phys. Radium (8) **16**, 21–38 (1955).

The author gives an extensive and detailed report on results obtained in 1949–1951 in solving equations

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho}-\Gamma^\sigma{}_{\mu\rho}g_{\sigma\nu}-\Gamma^\sigma{}_{\rho\nu}g_{\mu\sigma}=0$$

for the non-symmetric $\Gamma^\sigma{}_{\mu\nu}$ in terms of the non-symmetric $g_{\mu\nu}$ and its derivatives. These equations arise in Einstein's generalized theory of relativity. The methods given here are applied to the spherically symmetric case, both static and time-dependent. The results obtained in the former case are identical with those given by Bonnor [Proc. Roy. Soc. London. Ser. A. **210**, 427–434 (1952); MR **13**, 695].

A. H. Taub (Urbana, Ill.).

Hlavatý, Václav. The law of inertia in the unified field theory. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 153–167 (1954).

The asymmetric fundamental tensor $g_{\lambda\mu}$ of the unified field-theory uniquely defines a connection $\Gamma^\nu{}_{\lambda\mu}$. If four of the eighty-two equations of this theory are replaced by another set of four equations, a law of inertia results. This was obtained for special cases in earlier papers by the author [J. Rational Mech. Anal. **3**, 147–179, 645–689 (1954); MR **15**, 654; **16**, 408]. He now enunciates it for a general $g_{\lambda\mu}$ in the form: "A particle moving freely in the unified field $g_{\lambda\mu}$ describes a line, say C, which is autoparallel with respect to $\Gamma^\nu{}_{\lambda\mu}$". Thus, he says, the unified theory replaces the gravitational and electromagnetic forces by the unified field $g_{\lambda\mu}$, and he supports this statement by obtaining results that may be roughly described as follows: (1) To a first approximation the first three equations of C acquire at a point O the form of the classical Newtonian law of gravitation, while the fourth is identically satisfied at O; (2) to a second approximation the equations of C acquire at O the general-relativity (non-unified) form of Newton's second law of motion for an electromagnetic field. The meaning of 'approximation' is precisely defined. *H. S. Ruse.*

Bonnor, W. B. The equations of motion in the non-symmetric unified field theory. Proc. Roy. Soc. London. Ser. A. **226**, 366–377 (1954).

The Einstein unified field theory is described by 18 field equations. If one replaces the four ones which are of the third order (with respect to the unknowns $g_{\lambda\mu}$) by another set of four differential equations, then [see the paper reviewed above] the resulting "unified law of inertia" yields in the first approximation the Newton law of inertia and in the second approximation the Lorentz equations of motion. The author of the present paper modifies 14 equations (of the second and third order) of the theory, taking another

Hamiltonian. He shows that the resulting theory includes the Coulomb force in the equation of motion. The method used is based on the classical Einstein-Hoffmann-Infeld approach. *V. Hlavatý* (Bloomington, Ind.).

Hlavatý

★Riesz, Marcel. L'équation de Dirac en relativité générale. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 241–259 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The main objective is to deduce the Lorentz invariance of the Dirac equation, $(\nabla - ieA - im)\psi = 0$, and its solutions on a four-manifold endowed with a quadratic differential form equivalent pointwise to $dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$. The Clifford algebras associated with the tangent spaces are considered and in these terms fields of rotors and spinors are defined. The Dirac equation is interpreted into this context and the current-density vector field obtained from it is shown to have positive time component. The concepts involved can be more conveniently described in the language of fiber bundles. *A. M. Gleason* (Cambridge, Mass.).

Bhatt, M. P. A new form of line element for spherically symmetric solutions in general relativity. J. Maharaja Sayajirao Univ. Baroda **3**, no. 2, 119–123 (1954).

The author investigates spherically symmetric static solutions of the field equations, with energy-momentum tensor $\rho v^i v^k$, where v^k is null. He finds two non-diagonal solutions which he shows to be equivalent to the diagonal solution previously found by Vaidya [Proc. Indian Acad. Sci. Sect. A. **33**, 264–276 (1951); MR **13**, 391]. *F. A. E. Pirani.*

Mavridès, Stamatia. Sur une nouvelle définition du courant et de la charge en théorie unitaire d'Einstein. C. R. Acad. Sci. Paris **240**, 404–406 (1955).

The author considers two different possible interpretations of the field variable of Einstein's unified field theory. He then applies these interpretations to the spherically symmetric solutions of Papapetrou and the reviewer.

M. Wyman (Edmonton, Alta.).

Rosen, N. Some cylindrical gravitational waves. Bull. Res. Council Israel **3**, 328–332 (1954).

Einstein and the author investigated the general problem of cylindrical gravitational waves in space-time [J. Franklin Inst. **223**, 43–54 (1937)]. The author now considers some special cases. He finds standing wave and pulse wave solutions which are physically acceptable, but as he points out, a physical description of the source of the waves (in terms of the energy-momentum tensor) is still lacking.

F. A. E. Pirani (Dublin).

Mariot, Louis. Le champ électromagnétique singulier. C. R. Acad. Sci. Paris **239**, 1189–1190 (1954).

An electromagnetic field in general relativity, represented by a skew symmetric tensor, is singular if both the invariants of this tensor vanish. The author proves that if a field is singular on a spacelike hypersurface S, then it is also singular outside S. The proof makes use of Maxwell's equations and of the following facts established by H. S. Ruse [Proc. London Math. Soc. (2) **41**, 302–322 (1936)]: (i) for any electromagnetic field the energy tensor $\tau_{\alpha\beta}$ satisfies $\tau^{\alpha\rho}\tau_{\beta\rho} = k^2 g_\beta{}^\alpha$, where k is an invariant, and (ii) for a singular field $k=0$ and $\tau_{\alpha\beta} = P^2 l_\alpha l_\beta$, where P is an invariant and l_α a null vector. This null vector may be regarded as defining the world line of a photon [cf. L. Mariot, C. R. Acad. Sci. Paris **238**, 2055–2056 (1954); MR **15**, 995]. *J. L. Synge.*

Scherrer, Willy. Berichtigung und Ergänzung "Zur linearen Feldtheorie." Z. Physik **140**, 160–163 (1955).

A numerical correction is made to results given in the paper referred to in the title [Z. Physik **139**, 44–55 (1954); MR **16**, 635]. *A. H. Taub* (Urbana, Ill.).

Scherrer, Willy. Zur linearen Feldtheorie. II. Schwache Felder. Z. Physik **140**, 164–180 (1955).

The author assumes that various field quantities are such that their squares may be neglected. The field theory previously propounded [Z. Physik **138**, 16–34; **139**, 44–55 (1954); MR **16**, 79, 635] is then carried out in detail under this assumption. A class of fields is discussed which are said to correspond to light waves. Limitations on possible Lagrangeans from which the theory is devised are discussed.

A. H. Taub (Urbana, Ill.).

Galli, Mario. Osservazioni critiche circa nuove soluzioni del paradosso degli orologi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 356–363 (1954).

In relativity, two clocks, initially together and with the same readings, may show different readings if they are separated and brought together again. The author discusses various views that have been expressed on this so-called clock paradox, and gives his own. [The reviewer has never been able to see anything paradoxical in the fact that $\oint ds \neq 0$ for a closed circuit in space-time, which is what the so-called clock paradox amounts to mathematically.]

J. L. Synge (Dublin).

Ollendorff, F. A contribution to the treatment of the relativistic Keplerian motion. Bull. Res. Council Israel **3**, 25–30 (1953).

The author solves the classical Rutherford scattering problem, taking into account the relativistic mass increase of the scattered particle. *F. A. E. Pirani* (Dublin).

Wheeler, John Archibald. Geons. Phys. Rev. (2) **97**, 511–536 (1955).

An electromagnetic geon is a solution of the Einstein gravitational equations and the Maxwell equations in the absence of matter, of such a form that the energy is localized in a finite region of space and escapes from this region only very slowly. It is thus the nearest thing to a "body" that can be described purely within the domain of classical physics without using atomistic or quantum-mechanical notions. Two types of geon are studied in detail, one in the form of a torus with electromagnetic energy flowing round the "tube" in both directions, and one in the form of a sphere with electromagnetic waves randomly oriented inside it. The toroidal type is more stable because the gravitational field keeps the Maxwell field confined to the tube by a kind of total internal reflection. The spherical type is more amenable to exact quantitative calculation.

Classical physics can only be consistently applied so long as the electric field is less than the critical value $(m^2c^3/e\hbar)$ at which electron-positron pairs will spontaneously appear in the vacuum. This puts a lower limit of 10^{39} grams or 10^6 sun-masses on the possible mass of a classical geon. All masses larger than this limit are permissible. Similar conclusions also apply to "neutrino geons" consisting of large masses of neutrinos held together by gravitational forces.

F. J. Dyson (Princeton, N. J.).

de Castro, A. Relativistic cosmologies. Gaceta Mat. (1) **6**, 152–159 (1954). (Spanish)

Expository paper.

MECHANICS

Have

★Appell, Paul. Traité de mécanique rationnelle. Tome 5. Eléments de calcul tensoriel. Applications géométriques et mécaniques. Par René Thiry. 2ème éd. Gauthier-Villars, Paris, 1955. 202 pp. 3000 francs.

"Nouveau tirage" of the edition of 1933.

★Becker, Robert A. Introduction to theoretical mechanics. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. xiii+420 pp. $8.00.

This is a very good textbook for undergraduates in engineering physics. "The emphasis of the book is quite definitely toward the solution of problems". The surest way to cultivate an ability to do physical reasoning is to apply it." (From author's preface.) Many examples are worked out in the text. The book is arranged so that topics occur in the order of increasing difficulty. So after the more elementary mechanics of a particle, we meet the motion of a plane rigid body and then follows a chapter on central forces and Kepler's laws. In Appendices the necessary mathematical tools are given. Vectors are employed throughout the book. To give an idea of the scope of this introduction we mention that the last chapters deal with the motion of a rigid body in three dimensions, generalized coordinates, Lagrange's equations, vibrating systems, vibrating strings and wave motion. The book reflects modern trends: nonlinear systems, and those with varying mass are not omitted. It is very clearly written and may be highly praised for its scientific level and its instructive values. *O. Bottema.*

Sdobyrev, V. P. Criteria of stiffness of plane and spatial systems. Inžen. Sb. **15**, 187–190 (1953). (Russian)

The distribution of stresses in a framed structure is determinate if there are no redundant constraints. The addition of constraints makes the structure stiffer. This paper defines a criterion of stiffness in terms of the number of excessive constraints. This is expressed directly in terms of the types and the numbers of members and connections. Another expression of this criterion is in terms of the number of elementary regions in the graph of the structure and the reduced number of intersections. Examples are given as illustrations for both plane and spatial structures.

M. Goldberg (Washington, D. C.).

Kislicyn, S. G. A tensor method in the theory of spatial mechanisms. Trudy Sem. Teorii Mašin i Mehanizmov **14**, no. 54, 51–75 (1954). (Russian)

This, rather obscure, paper may very well be a turning point for the now languishing space-linkage theory. Without any new theoretical results, it demonstrates that line geometry is not a toy but a tool (if not the tool) for space-linkage study. The author rediscovers old matter [cf., e.g., W. Blaschke, Vorlesungen über Differentialgeometrie, vol. I, 3rd ed., Springer, Berlin, 1924, p. 266] which goes back to E. Study [Geometrie der Dynamen, Teubner, Leipzig, 1903, p. 205] or to A. P. Kotelnikov [the unobtainable Calculus of screws, (in Russian), Kazan, 1895], both unquoted. The simple idea consists in representing a rigid motion as a line-to-line transformation, the lines being represented by Clifford numbers, $a+ob$, $o^2=0$ [cf., e.g., Blaschke, loc. cit., p. 262, or L. Brand, Vector and tensor analysis, Wiley, New York, 1947, p. 73 [MR **9**, 68] where they are called "dual" numbers]. These numbers were applied to space linkages before by F. M. Dimentberg [Trudy Sem. Teorii Mašin i Mehanizmov **5**, no. 17; 5–39 (1948); MR **12**, 549] but in a different way, using the Cliffordian vector calculus. In this paper, however, a rigid motion is represented by a three-by-three matrix of Clifford ("dual") cosines, in exact imitation of the cosine matrix in motion about a fixed point. Implicitly, the essence of the (as yet unproved) Study-Kotelnikov "principle of extension" (cf., e.g., Blaschke, loc. cit., p. 264) is used to obtain the matrix in a few lines. The proof of the pudding is the derivation of purely scalar equations of performance, linking the position parameters of a fairly complicated mechanism whose essential component contains six rigid members linked, consecutively, as follows: $HUSU'H'U''$ (H=turning pair, S=sliding pair, U=universal joint) where the axis of H goes through the center of U, the axis of S is parallel to the line joining the centers of U and U', and the axis of H' goes through the center of U'. The reader should not share the author's embarrassment over the fact that 12 (the author's says "18") equations arise for the six angular and linear variables involved: the additional six equations bind the linear variables to the angular ones (in accord with the principle of extension) and are the "compatibility conditions" necessary for the assembly of the mechanism: in a nonsingular spatial mechanism the angular relations determine the changes of the linear parameters.

A. W. Wundheiler (Chicago, Ill.).

Gale, E. I. Accessory linkages which have certain stabilizing properties. Amer. Math. Monthly **62**, 94–99 (1955).

Two different linkage mechanisms may describe identical curves. However, each may have a position of instability (a dead-center) at which there is an ambiguity of possible subsequent motion. If these two dead-centers occur at different parts of the curve, the two mechanisms can be combined into one without a position of instability. As examples of the foregoing, the Hart cell and the Peaucellier cell are both inversors. When used in describing an elliptic cissoid, only one-half of a Hart cell need be added to a Peaucellier cell to eliminate the instability. Other examples given are combinations of inversors to describe hyperbolas, lemniscates and limaçons.

Other methods which have been used to eliminate dead-centers are duplicate mechanisms which are placed out-of-phase and parts of toothed gear wheels. *M. Goldberg.*

Aržanyh, I. S. On the application of functions of a complex variable to the dynamics of a material point. Dokl. Akad. Nauk SSSR (N.S.) **96**, 21–24 (1954). (Russian)

This paper presents several theorems on existence of integrals of dynamical systems. The theorems and their proofs remain incomprehensible to the reviewer. The first theorem states: "The system of differential equations $\ddot{x}=U_x+2\omega\dot{y}$, $\ddot{y}=U_y-2\omega\dot{x}$ has always a linear integral (1) $S\dot{x}+T\dot{y}=K$." The context makes clear that S, T, K are to be functions of (x, y) and that the energy constant is to be assumed fixed. The condition that (1) holds on solution curves then leads to a complicated non-linear first-order partial differential equation for K and to the Cauchy-Riemann equations for S, T. At this point, the proof stops. If $S+iT$ is indeed analytic and K can be found from its equation, then one has at best an invariant relation and not an integral. The second and third theorem contain similar assertions and proofs concerning integrals quadratic in $\dot{x}$, $\dot{y}$.

W. Kaplan (Ann Arbor, Mich.).

Šul'gin, M. F. On integration of the dynamical equations of S. A. Čaplygin. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. i Meh. 5, 119–128 (1949). (Russian)

The paper deals with the Čaplygin case of nonholonomic constraints in which $\dot{q}^i = B_\lambda{}^i \dot{q}^\lambda$ $(i, j, =1, \cdots, m;$ $\lambda, \mu = m+1, \cdots, n$; sum over repeated indices), and $B_\lambda{}^i$, T, and V do not depend on q^i. In familiar (although not the author's) notation, let us put

$$\Omega^i{}_{\lambda\mu} = \partial_\mu B_\lambda{}^i - \partial_\lambda B_\mu{}^i, \quad p_i = \partial T/\partial \dot{q}^i, \quad p_\lambda = \partial T/\partial \dot{q}_\lambda + B_\lambda{}^i p_i,$$
$$\dot{q}^\lambda p_\lambda - T + V = H(q^\lambda, p_\lambda, t).$$

Then the equations of motion are

$$\dot{q}^\lambda = \partial H/\partial p_\lambda, \quad \dot{p}_\lambda = -\partial H/\partial q^\lambda + \Omega^i{}_{\lambda\mu} \dot{q}^\mu p_i.$$

If S is a function of q^N $(N=1, \cdots, n)$ and n variables a_N, satisfying the equations

$$\partial S/\partial t + H(q^\lambda, p_\lambda, t) = 0, \quad p_\lambda = \partial S/\partial q^\lambda + B_\lambda{}^i \partial S/\partial q^i,$$

then $\partial S/\partial a_N = b^N$ are integrals of the equations of motion for every set of constants a_N, b^N compatible with the constraints. *A. W. Wundheiler* (Chicago, Ill.).

Percev, B. P. The theory of Pošehonov's pendulum Astr. Ž. 31, 90–96 (1954). (Russian)

The pendulum in question, mounted in the Moscow planetarium, has a horizontal oscillation axle which can rotate about a vertical axis. Its motion relative to the earth is studied with the aid of Lagrange's equations. For small amplitudes, the two parameters (neglecting friction) can be expressed as elliptic integrals of the time. The instrument, given proper initial conditions, has a vertical angular velocity about twenty times of that of the Foucault pendulum, and requires little demonstration space. The influence of friction is claimed to be slight. *A. W. Wundheiler.*

***Mazet, Robert. Mécanique vibratoire.** Librairie Polytechnique Ch. Béranger, Paris et Liége, 1955. xix+280 pp. 4975 francs.

Except for the absence of problems and an index, this book might well serve as a text for a senior or beginning graduate course in vibration theory. Primarily it is a collection in one place of known results in the theory of vibrating systems of from simple type to ones of intermediate complexity. The mathematical techniques are standard for this level of engineering training.

The first chapter is introductory. The second chapter treats conservative systems of one degree of freedom in substantial detail. The third chapter treats dissipative systems of one degree of freedom in great detail. The fourth chapter treats conservative systems having two degrees of freedom. Coupling is emphasized and many physical examples are discussed. In the fifth chapter, "chains" are discussed. These are systems composed of a number of elements which are arranged in a sequence such that each member couples only with its two neighbors. Examples are filters and transmission lines, and automobile crankshafts. In the sixth chapter dissipative systems of several degrees of freedom are discussed, including two pendulums with a friction coupling and several other mechanical and electrical examples. Chapter seven treats forced vibrations supported by external energy sources, such as in reeds in wind instruments, and in airplane flutter. All systems considered in the first seven chapters are linear or conservative systems having an integral. The eighth chapter, however, briefly discusses some of the implications of nonlinearity in vibrating systems. The format is good, and the collection of physical results is valuable. *E. Pinney* (Berkeley, Calif.).

García, Godofredo. On the gyroscopic motion of a projectile. Actas Acad. Ci. Lima 17, 51–65 (1954). (Spanish)

Hunziker, Raúl Ricardo. Effect of the wind on the active trajectory of a rocket. Univ. Nac. Eva Peron. Publ. Fac. Ci. Fisicomat. no. 206, Serie Tercera. Publ. Esp. 43, 54–67 (1953). (Spanish. English summary)

The author starts with the differential equations of the active trajectory (that before burn-out) of a rocket, expressed in great generality, including, in particular, effects of constant wind. By a succession of simplifying assumptions, and by simultaneous consideration of several trihedra of reference, he arrives at separated closed expressions (in terms of the drag coefficient) for the approximate direction angles of the velocity vector and displacements at burn-out (initial conditions for the "coasting" trajectory). He concludes with six qualitative and quantitative inferences concerning the effective drag in terms of varying initial data. *A. A. Bennett* (Providence, R. I.).

Hydrodynamics, Aerodynamics, Acoustics

Reiner, M. On volume-viscosity. Bull. Res. Council Israel 3, 67–71 (1953).

The author is concerned with materials for which the stress is a function of a tensor measuring strain associated with "the recoverable part of the deformation" and a tensor measuring rate of deformation associated with "irrecoverable displacement." His main purpose seems to be to determine how such a material will respond to hydrostatic stress. He does not indicate how one gets sufficient equations to determine both "recoverable" and "irrecoverable" displacements. On the basis of several assumptions, he concludes that the response of such a material to hydrostatic stress can be suitably represented by a mechanical model composed of springs and dashpots. There are errors in the meagre analysis given, e.g., his equation (3.4) is, in general, wrong if, as is stated on p. 67, the "elastic bulk modulus" depends on strain. The paper also suffers from a lack of precise definitions. *J. L. Ericksen* (Washington, D. C.).

Tyabin, N. V. Theory of the anomalous viscosity of dispersive systems. Ž. Tehn. Fiz. 25, 339–350 (1955). (Russian)

Müller, W. Zur Bestimmung der Trägheitskoeffizienten unsymmetrischer Rotationskörper. Österreich. Ing.-Arch. 8, 263–284 (1954).

If a uniform stream in the direction AB is disturbed by a point source at A and an equal uniform line sink stretching from A to B, the dividing stream surface consists in part of a closed surface S of revolution about AB. The author gives tables for drawing the meridional section. Continuing a paper [same Arch. 8, 171–184 (1954); MR 16, 82] he approximates to the moment when a solid bounded by S is placed at incidence in a stream. Results are compared with those for a spheroid of the same length and volume. *L. M. Milne-Thomson* (Greenwich).

De, Kamini Kumar. Resistance on an infinite cylinder due to two-dimensional motion past the cylinder, of a fluid having uniform vorticity. Bull. Calcutta Math. Soc. 46, 81–85 (1954).

The author applies his method [same Bull. 45, 121–124 (1953); MR 15, 754] to the cylinder η=constant in the net

$$z=nae^{-i\zeta}+be^{in\zeta}\quad (b\leqq a).$$

L. M. Milne-Thomson (Greenwich).

Prem Prakash. Superposition of orthogonal flows. Math. Student 22, 129–135 (1954).

This paper is concerned with mutually orthogonal and superposable flows of an incompressible viscous fluid, that is, flows for which the velocity fields (u_1, v_1), (u_2, v_2), and (u_1+u_2, v_1+v_2) are all solutions of the Navier-Stokes equations, and for which also $u_1u_2+v_1v_2=0$. Specializing to such flows of the form $(u_1, 0)$ and $(0, v_2)$, and also $(u_r, 0)$, $(0, v_\theta)$ (in polar coordinates), the author finds the possible flows to be certain known solutions of the Navier-Stokes equations, the solutions in the former case, for example, representing the decay of vorticity between straight parallel walls.

D. Gilbarg (Stanford, Calif.).

Nigam, S. D., and Chatterji, P. P. Hydrodynamical equations for the motion of bodies of revolution in non-viscous rotating liquid. Quart. J. Mech. Appl. Math. 7, 458–461 (1954).

A linear differential equation is derived for the stream function of a steady, axisymmetric flow in a rotating fluid in terms of any orthogonal, curvilinear, coordinate system having the azimuthal angle about the axis of rotation as one coordinate. This is a generalization of the equation previously derived by Long [J. Meteorol. 10, 197–203 (1953)] who restricts himself to spherical and cylindrical systems.

G. Morgan (Providence, R. I.).

Iglisch, Rudolf. Elementarer Beweis für die Eindeutigkeit der Strömung in der laminaren Grenzschicht zur Potentialströmung $U=u_1x^m$ mit $m\geqq 0$ bei Absaugen und Ausblasen. Z. Angew. Math. Mech. 34, 441–443 (1954). (English, French and Russian summaries)

In a recent paper [same Z. 33, 143–147 (1953); MR 15, 262] the author showed by elementary methods the existence of a solution of the boundary layer equation $f'''+ff''+\beta(1-f'^2)=0$ under the boundary conditions $f(0)=C$, $f'(0)=0$, $f'(\infty)=1$; and satisfying the side condition $f'(\eta)>0$, $0<\eta<\infty$; the constant C may be arbitrary while $\beta>0$. The above equation is the boundary-layer equation for flow over a wedge-shaped body with flare angle $\beta\pi$. In this paper, it is proven by elementary methods, using the differential equation, that the solution of the above problem is unique for $\beta\geqq 0$. For the particular case $\beta=0$ it is necessary that $C>-1.2385\cdots$ in order that a solution exist.

R. C. DiPrima (Boston, Mass.).

Berlyand, O. S. On approximate solution of some problems of the hydrodynamics of the boundary layer. Trudy Geofiz. Inst. no. 22(149), 127–130 (1954). (Russian)

Mirels, Harold. Laminar boundary layer behind shock advancing into stationary fluid. NACA Tech. Note no. 3401, 25 pp. (1955).

L'étude entreprise dans cet article précise l'influence de la viscosité sur le mouvement d'une onde de choc plane (perpendiculaire à $0x$) qui se déplace parallèlement à un mur plan $(y=0)$, dans un fluide en mouvement stationnaire. La viscosité est supposée proportionnelle à la température absolue T. L'intégration des équations de la couche limite est ramenée à l'intégration de plusieurs équations différentielles de la façon suivante; on choisit une variable η proportionnelle à $x^{-1/2}\int_0^y T^{-1}dy$ et on suppose la fonction de courant proportionnelle à $x^{1/2}f(\eta)$; on obtient ainsi pour la fonction f l'équation différentielle $f'''+ff''=0$. Des tables numériques sont construites pour six valeurs différentes de l'intensité du choc; les résultats concernent la vitesse et la température du fluide, ils intéressent les utilisateurs des tubes de choc.

H. Cabannes (Marseille).

Michael, D. H. A two dimensional magnetic boundary layer problem. Mathematika 1, 131–142 (1954).

In this paper two-dimensional hydromagnetic problems are considered of which the following is typical. An infinite cylinder which is a perfect electrical conductor moves through a fluid of finite electrical conductivity (σ) and kinematic viscosity (ν) and encounters a magnetic field distribution which is initially normal to the axis of the cylinder; and the problem concerns the manner of the redistribution of the magnetic field which takes place. If the magnetic diffusivity λ $(=1/4\pi\sigma\mu$ where μ denotes the magnetic permeability) is large compared to ν, then what happens is effectively the formation of a magnetic boundary layer around the cylinder. The nature of a resulting boundary layer is studied. A theorem of general interest which is established is that under the circumstances considered, the magnetic flux per unit length which crosses the path of the cylinder remains constant throughout the motion.

S. Chandrasekhar (Williams Bay, Wis.).

***Lin, C. C. On periodically oscillating wakes in the Oseen approximation.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 170–176. Academic Press Inc., New York, 1954. $9.00.

The oscillating vortex wake behind an obstacle at Reynolds numbers of order 10^3 is studied by means of the Oseen approximation. *M. J. Lighthill* (Manchester).

Lin, C. C. Some physical aspects of the stability of parallel flows. Proc. Nat. Acad. Sci. U. S. A. 40, 741–747 (1954).

This paper deals with the distribution of the Reynolds stresses in the problem of the hydrodynamic stability of plane-parallel flows.

Given a basic flow (U) parallel to a plane boundary and considering a disturbance which results in the velocities $(U+u, v)$ then one generally expresses

$$u=\Re\{f(y)e^{i(\alpha x-\omega t)}\},\quad v=\Re\{g(y)e^{i(\alpha x-\omega t)}\},$$

where α and ω are constants and $f(y)$ and $g(y)$ represent the expansions

$$(1)\quad f(y)=u_1y+u_2y^2+\cdots;\quad g(y)=v_1y+v_2y^2+v_3y^3+\cdots.$$

From the equation of continuity it follows that

$$v_1=0,\quad 2v_2=-i\alpha u_1,\quad 3v_3=-i\alpha u_2.$$

The corresponding Reynolds stress τ is given by

$$\tau=-\tfrac{1}{2}\rho\alpha y^4\Re(iu_1^*u_2).$$

Using the foregoing formula, we find

$$\tau=-\frac{2}{3\alpha}\Re(iu_2/u_1)\rho\overline{v^3},$$

where $\overline{v^2}=\frac{1}{2}\Re(gg^*)$ is the mean-square value of the v-com-

ponent of the disturbance. From the known solution of the distribution of vorticity ζ in an oscillating stream over a stationary plate, the author now supposes that

$$\zeta = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = Ae^{i(\alpha x - \omega t)}e^{-(1-i)\eta}, \tag{2}$$

where A is a constant

$$\eta = y/\delta_0 \quad \text{and} \quad \delta_0 = (2\nu/\omega)^{1/2}. \tag{3}$$

From equations (1), (2) and (3) it now follows that

$$\frac{u_2}{u_1} = -\frac{1-i}{2\delta_0} \quad \text{or} \quad \Re\left(\frac{iu_2}{u_1}\right) = -\frac{1}{2\delta_0}.$$

Hence $\tau = \rho\overline{v^2}/3\alpha\delta_0$ which is always positive.

Next, considering the distribution of the Reynolds stress away from the wall, the equation of motion (neglecting viscosity) for the vorticity

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} - \left(\frac{d^2U}{dy^2}\right)v = 0$$

is solved for v when $\zeta = Ae^{\lambda t}\sin(\alpha x - \omega t + \epsilon)$, where A and ϵ are functions of y. The Reynolds stress is then computed from the formula $d\tau/dy = \rho\overline{v\zeta}$. In this manner the author obtains

$$\frac{d\tau}{dy} = \rho\overline{v^2}\frac{c_iU''/\alpha}{(U-c_r)^2 + c_i^2} \quad (c_r = \omega/\alpha;\ c_i = \lambda/\alpha).$$

From this equation it follows that $(\tau/\rho\overline{v^2})$ has a jump at the critical layer where the flow speed equals the wave speed. The discontinuity is

$$\left[\frac{\tau}{\rho\overline{v^2}}\right] = \pm\frac{\pi}{\alpha}\left[\frac{d^2U/dy^2}{dU/dy}\right]_{\text{critical layer}}.$$

From this last result Rayleigh's theorem (that a neutral oscillation must have a wave speed equal to that of the basic flow at the point of inflection) is deduced.

By combining the foregoing discussion applicable near the wall and away from the wall respectively, the author examines the distribution of the Reynolds stresses over the whole region and shows how these elementary considerations are related to the basic mathematical theory of hydrodynamic stability of plane-parallel flows.

S. Chandrasekhar (Williams Bay, Wis.).

Bass, J. Space and time correlations in a turbulent fluid. II. Space and time spectrum. Univ. California Publ. Statist. **2**, 85–98 (1954).

[For part I see same Publ. **2**, 55–83 (1954); MR **15**, 908.] The author states that this paper is an introduction to the study of the general spectrum of turbulence, considering velocity correlations at two different instants of time. Only the geometrical aspects of the question are investigated; the dynamical properties of the spectrum are not considered. The analysis is quite thorough, and should therefore provide useful background for further work. Some limited comparison with experiments is given.

C. C. Lin.

Lehnert, B. The decay of magneto-turbulence in the presence of a magnetic field and Coriolis force. Quart. Appl. Math. **12**, 321–341 (1955).

It is known that during the last stage in the decay of isotropic turbulence the energy density in a wave number k decreases like $\exp\{-2\nu k^2t\}$ where ν denotes the kinematic viscosity. In the first part of this paper the author investigates the manner of decay when an external magnetic field (**H**) is impressed and we are dealing with turbulence in an electrically conducting fluid. By introducing the corresponding spectral tensors describing the correlations in the velocity components and the magnetic field intensities at two different points, the author shows that the decay involves the three characteristic times

$$\tau_{12}^{-1} = k^2[(\lambda+\nu) \pm (\lambda-\nu)(1-\zeta^2)^{1/2}], \quad \tau_{13}^{-1} = k^2(\lambda+\nu),$$

where $\lambda = 1/(\mu\sigma)$ (σ denoting the coefficient of electrical conductivity in MKSA-units) and

$$\zeta = 2\left(\frac{k^2H^2\mu\cos^2\vartheta}{\rho}\right)^{1/2}\frac{1}{k^2(\lambda-\nu)},$$

where H is the strength of the impressed field, ϑ the inclination of the direction of the wave vector **k** and **H**, μ the coefficient of magnetic permeability and ρ the density. For conducting liquid metals (such as mercury) $\lambda \sim 10^6\nu$ and the spectral tensor Ω_{ij} describing the correlation in the components of the velocity behaves like

$$\Omega_{ij}(\mathbf{k}, t) \approx \Omega_{ij}^{(1)}(\mathbf{k})\exp\left\{-2\left[\nu k^2 + \frac{k^2H^2\mu\cos^2\vartheta}{\rho(\lambda-\nu)}\right]t\right\}.$$

From this equation the author concludes that during the decay of turbulence in a liquid with $\lambda \gg \nu$ all periodic turbulence elements as well as the aperiodic ones with small extensions in the direction of the field (large values of $k\cos\vartheta$) are damped out relatively rapidly; further, that the asymptotic state of decay is two-dimensional with respect to the direction of the field. In terms of these results an explanation is found for the experimentally observed inhibition of turbulence in mercury by a magnetic field.

In the second part of the paper the same problem is investigated when in addition to a magnetic field (in the yz-plane) the fluid is subject to rotation with an angular velocity Ω about the z-axis. In this case the decay of the spectral tensors with indices ij involves four reciprocal characteristic times given by

$$m = -(a+b) \pm (2)^{-1/2}\{c^2 - 4F^2 - 4\Omega^2\epsilon_{ij3}'^2 \pm [(c^2-4F^2)^2 + 8\Omega^2\epsilon_{ij3}'^2(c^2+4F^2+2\Omega^2\epsilon_{ij3}'^2)^2]^{1/2}\}^{1/2},$$

where $(a+b) = k^2(\lambda+\nu)$, $c = k^2(\lambda-\nu)$, $F = kH\cos\vartheta(\mu/\rho)^{1/2}$, and ϵ_{ij3}' is the usual alternating tensor except that it refers to the off-diagonal terms ($i \neq j$) only. The physical consequences of these decay times are discussed and illustrated.

S. Chandrasekhar (Williams Bay, Wis.).

✱Ward, G. N. Linearized theory of steady high-speed flow. Cambridge, at the University Press, 1955. xv+243 pp. $6.00.

Ce livre est un exposé de la théorie linéarisée des écoulements à grande vitesse. La première partie est un exposé général; les équations du mouvement et les conditions aux limites sont successivement linéarisées; le fait que la pression est indépendante des composantes du tourbillon normales aux lignes de courant est mis en relief [Lighthill, Phil. Mag. (7) **40**, 214–220 (1949); MR **10**, 641]. La solution générale des equations linéarisées est ensuite indiquée dans le cas des écoulements subsoniques et des écoulements supersoniques. Dans le second cas les propriétés des tourbillons en fer à cheval sont établies. A une grande distance de l'obstacle la théorie linéarisée ne donne pas de résultats satisfaisants; la méthode imaginée par Lighthill et appliquée par Witham [Comm. Pure Appl. Math. **5**, 301–348 (1952); MR **14**, 330] pour éviter cet inconvénient est brièvement signalée.

La seconde partie contient la solution de quelques problèmes particuliers; écoulement subsonique autour d'un obstacle aplati, écoulement supersonique autour d'une aile peu epaisse, écoulements coniques. Les écoulements supersoniques sont étudies plus particulièrement à l'aide de la transformation de Laplace. La troisième partie est consacrée à l'étude de l'écoulement au voisinage d'un corps de révolution élancé; après avoir établi l'expression du potentiel linéarisé, l'auteur donne les formules qui déterminent les forces aérodynamiques. Le cas d'un obstacle dont la méridienne possède un point anguleux est également traité. Ce livre qui sera très utile pour tous ceux qui étudient l'aérodynamique, est remarquable par la richesse des résultats et par la clarté de l'exposé. *H. Cabannes* (Marseille).

Gilbarg, David, and Serrin, James. Uniqueness of axially symmetric subsonic flow past a finite body. J. Rational Mech. Anal. **4**, 169–175 (1955).

Dans le cas des mouvements à symétrie axiale, les Auteurs établissent l'unicité, pour un nombre de Mach à l'infini donné, de l'écoulement subsonique autour d'un obstacle. Pour les mouvements plans, ce problème d'unicité avait été traité par Bers [Comm. Pure Appl. Math. **7**, 441–504 (1954); MR **16**, 417], au moyen de la méthode de l'hodographe et, d'une manière élémentaire, par Gilbarg [J. Rational Mech. Anal. **2**, 233–251 (1953); MR **14**, 920], en utilisant le principe du maximum pour une équation aux dérivées partielles du type elliptique. Cette dernière méthode est étendue ici aux mouvements à symétrie axiale. L'unicité s'obtient très simplement à partir d'un théorème de Phragmén-Lindelöf que les Auteurs démontrent pour une équation elliptique de la forme

$$Au_{xx}+2Bu_{xy}+C\left(u_{yy}-p\frac{u_y}{y}\right)+Du_x=0$$

$$(p>0,\ yD\to 0 \text{ à l'infini}).$$

R. Gerber (Toulon).

***Loewner, Charles. Some bounds for the critical free stream Mach number of a compressible flow around an obstacle.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 177–183. Academic Press Inc., New York, 1954. $9.00.

En utilisant des résultats d'un travail antérieur [Loewner, J. Rational Mech. Anal. **2**, 537–561 (1953); MR **15**, 73] l'Auteur construit une borne supérieure pour le nombre de Mach critique d'un écoulement plan autour d'un obstacle. La borne ainsi construite ne dépend que de l'obstacle et non de l'équation d'état du fluide. *R. Gerber* (Toulon).

Dubinskiĭ, M. G. On rotating gas flows. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 8, 75–78 (1954). (Russian)

In a swirling axisymmetric flow with non-zero velocity components c_z=const. parallel to the axis of symmetry of a system of cylindrical coordinates r, z, and $c_u=\Omega r+\Delta(r)$ normal to planes through the z-axis, where Ω is constant, the pressure $p(r)$, density $\rho(r)$, and specific entropy $s(r)$ are subject to only two conditions $dp/dr=\rho c_u^2/r$ and the adiabatic law. In such a flow with $\Delta(r)=0$ the pressure $p_1=p(R)$ at the surface of a tube of radius R and $p_0=p(0)$ on its axis can be expressed in terms of the rate of flow across any section z=const. of mass, momentum, angular momentum, kinetic energy, and E_u the kinetic energy due to swirl. If to obtain a third condition for p, ρ, and s the function $p(r)$ is required to be such as to produce maximal rate of entropy flow with constant E_u, an isoperimetric problem results whose solution implies $(p-p_0)/(\rho-\rho_0)$=const. This relation can be satisfied in particular by the radial temperature distribution $T(r)$=const. Finally, of all the swirling flows with given Ω and density distribution $\rho(r)$ the choice $\Delta(r)=0$ yields minimum E_u. *J. H. Giese.*

Hida, Kinzo. Supplementary note to my paper "An approximate study on the detached shock wave in front of a circular cylinder and a sphere." J. Phys. Soc. Japan **10**, 79–81 (1955).

See same J. **8**, 740–745 (1953); MR **15**, 838.

Keller, Joseph B. Geometrical acoustics. I. The theory of weak shock waves. J. Appl. Phys. **25**, 938–947 (1954).

The fundamental equations of continuum mechanics and the various jump conditions across different types of discontinuity surfaces are derived for an arbitrary continuous fluid. A perturbation (called the acoustic flow) of a continuous solution of these equations (called the basic flow) is considered and the fundamental (linearized) equations and various jump conditions across an arbitrary discontinuity surface of the acoustic flow are derived. These equations are specialized to a perfect non-heat-conducting fluid and the case where the discontinuity surface of the acoustic flow corresponds to a weak shock wave is considered in detail. A first-order partial differential equation which determines the weak shock front in terms of the basic flow is found. This equation occurs in geometrical optics and following classical theory the problem is now considered in terms of wave fronts and characteristic curves. Results are found for the variation of the strength of a weak shock along a characteristic and for the reflection and transmission of such a shock at a contact discontinuity of the basic flow. A detailed analysis of an "acoustic" shock tube is also made. *P. Chiarulli.*

Mackie, A. G., and Pack, D. C. Transonic flow past finite wedges. J. Rational Mech. Anal. **4**, 177–199 (1955).

Les auteurs complètent notablement dans le présent article les résultats obtenus dans une étude antérieure [Proc. Cambridge Philos. Soc. **48**, 178–187 (1952); MR **13**, 701]. Dans une première partie est rappelée la solution proposée pour l'étude de l'écoulement transsonique autour d'un obstacle en coin placé symétriquement dans l'écoulement. L'analyse se développe à partir de l'équation exacte de Chaplygin écrite dans le plan de l'hodographe; le point de départ est le développement en série de la solution en incompressible au voisinage du point de vitesse nulle; par un procédé classique on peut associer à cette solution une solution correspondant à un écoulement compressible qui vérifie certaines conditions aux limites du problème (en gros celles qu'il convient d'écrire dans la région subsonique du plan de l'hodographe). Une telle solution dépend en fait du choix fait pour une fonction arbitraire. L'interprétation et la discussion des singularités susceptibles d'apparaitre font l'objet d'une étude détaillée; une comparaison avec les solutions connues dans le cas où l'angle du coin est faible [e.g. J. D. Cole, J. Math. Phys. **30**, 79–92 (1951); MR **15**, 263] montre que, pour des hypothèses simplificatrices comparables, la méthode proposée a l'avantage de s'appliquer à des cas plus étendus.

La deuxième partie envisage de façon systématique le calcul de la trainée pour la classe des approximations proposées. Celui ci se présente de façon élégante; dans le cas des angles faibles le résultat s'exprime de façon simple à

l'aide de la fonction $\zeta(s)$ de Riemann. Une comparaison avec le résultat classique de Guderley et Yoshihara [J. Aero. Sci. **17**, 723–735 (1950); MR **15**, 264] (qui peut être considéré comme exact dans le cas des angles petits lorsque l'écoulement non perturbé est juste sonique) révèle une erreur relative de 12% environ, par défaut. Dans le cas où l'angle du coin est égal à 90°, c'est à dire d'une plaque plane normale au vent, le résultat prend une forme simple; le résultat obtenu est en excellent accord avec le résultat obtenu par Maccoll en utilisant une méthode semi empirique.

P. Germain (Paris).

Tricomi, Francesco G. Un viaggio attraverso il muro del suono. Confer. Sem. Mat. Univ. Bari no. 3, 14 pp. (1954).

Il s'agit d'une conférence, tout à fait semblable à celles déjà analysées dans ces revues [Univ. e Politec. Torino. Rend. Sem. Mat. **12**, 37–52 (1953); Monatsh. Math. **58**, 160–171 (1954); MR **15**, 661; **16**, 536]. *P. Germain.*

Ludloff, H. F., and Friedman, M. B. Aerodynamics of blasts—diffraction of blast around finite corners. J. Aero. Sci. **22**, 27–34 (1955).

The motion of a strong shock parallel to a wall up to and beyond a corner or edge at right angles to the direction of propagation, where the wall turns through an arbitrary angle, is considered in detail with special emphasis on corners concave to the flow and angles such that Mach reflexion occurs. Two approaches to the problem of calculating the field of flow are outlined.

One approach ("the elliptic method") makes use of the fact that the flow quantities must be functions of $\xi=x/t$ and $\eta=y/t$ alone, where x, y are Cartesian co-ordinates with origin at the corner and t is time measured from when the shock wave first reaches the corner. The differential equations of flow without viscosity and heat conduction are written down using ξ and η as co-ordinates [following Jones, Martin and Thornhill, Proc. Roy. Soc. London. Ser. A. **209**, 238–248 (1951); MR **13**, 1001]. Next, conditions in the vicinity of the singular point on the wall (beyond the corner), where the various streamlines or "isentrops" meet, are treated analytically by a process of series expansion. (This point is observed in photographs as the point where the "slipstream" or vortex sheet from the three-shock intersection meets the wall.) Arbitrary constants in the series expansion are given values such that the resulting pressure distribution along the wall agrees with experimental results [taken from Bleakney and White, N6ori-105, Tech. Rep. II-10 (1951)]. It is stated that an alternative approach will be presented elsewhere, in which the equations of motion, together with the boundary conditions at the (initially unknown) positions of the slipstream and shock waves, will be treated by a relaxation process, in which of course the analytical character of the solution near the singular point is an essential datum.

A second approach ("the hyperbolic method") makes use of P. Lax's discovery [see, e.g., Comm. Pure Appl. Math. **7**, 159–193 (1954); MR **16**, 524] that if the equations of motion are put into "conservation" form (equating the time derivatives of the mass, momentum and energy densities respectively to minus the divergences of the mass, momentum and energy flux vectors), and then approximated by a stable finite-difference scheme, then shock regions of the order of a few grid mesh lengths in thickness occur in the solution approximately where they would in the real gas. (The use of an appropriate finite-difference scheme introduces an effective viscosity, and the satisfaction of the Rankine-Hugoniot equations at the shock is assured because the form of equations forces overall conservation of mass, momentum and energy.) The authors find that when this approach is applied to their problem they obtain the characteristic Mach reflexion pattern already after five multiples of their basic time interval. A much greater number will be required to reach a state accurately descriptive of the flow field as a whole, and the work has not yet been carried to this stage. The authors suggest that the later stages in the process might most conveniently be treated by using ξ, η and t as co-ordinates instead of x, y, and t, but it remains to be seen whether Lax's result will still apply when this has been done. *M. J. Lighthill.*

Brescia, Riccardo. Studio dell'interferenza delle gallerie aerodinamiche con pareti a fessure in correnti compressibili. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **88**, 79–106 (1954).

The author's earlier work [same Atti **87**, 225–244 (1953); MR **16**, 304] on interference in incompressible flow in slotted-wall wind tunnels is extended to subsonic compressible flow by means of the Prandtl-Glauert correction. Under the resulting geometrical distortion some approximations in the original paper may become invalid. If this happens, the author suggests various improvements to the earlier results, such as truncation later in a series, approximation of a wing in the tunnel by more than one point singularity, etc. Tables are presented for use in determining as functions of model size, tunnel width, and Mach number the wall interference corrections to angle of attack and slope of the lift-coefficient curve, and the gap-to-slat ratio of the wall to cancel interference at a given point on a wing.

J. H. Giese (Havre de Grace, Md.).

Nonweiler, T. Theoretical supersonic drag of non-lifting infinite-span wings swept behind the Mach lines. Aeronaut. Res. Council, Rep. and Memo. no. **2795** (1950), 22 pp. (1954).

This paper is concerned with the wave drag of a constant-chord swept-back wing with straight subsonic leading edge and constant symmetrical wing sections at zero incidence. For the case of infinite span, the author obtains an elegant and compact expression for the wave drag, which is reminiscent of the formula for the induced drag in lifting-line theory. It is shown that, for given sweepback, the wave drag can be represented as the product of two terms which are, respectively, independent of the flight speed and of the wing section. Various examples are given and the dependence of the drag on certain parameters is discussed in detail.

A. Robinson (Toronto, Ont.).

Martin, John C., and Gerber, Nathan. The effect of thickness on airfoils with constant vertical acceleration at supersonic speeds. J. Aero. Sci. **22**, 179–188 (1955).

Van Dyke's iteration procedure is applied to the problem of the title. The authors' results have been obtained previously by Van Dyke [same J. **20**, 61 (1953); NACA Tech. Note no. **2982** (1953); MR **16**, 419]. *J. W. Miles.*

Garner, H. C. Methods of approaching an accurate three-dimensional potential solution for a wing. Aeronaut. Res. Council, Rep. and Memo. no. **2721** (1948), 19 pp. (1954).

This is a short review of some of the approximate procedures available for calculating the lift distributions of thin

lifting surfaces. Proposals are made for calculations, including iterative methods, which might be used to cast light on the accuracies of these various approximations.

W. R. Sears (Ithaca, N. Y.).

Woods, L. C. On unsteady flow through a cascade of aerofoils. Proc. Roy. Soc. London. Ser. A. **228**, 50–65 (1955).

The case considered is that of an infinite, unstaggered cascade of identical thin airfoils. After transforming the plane of these airfoils by means of a conformal transformation into a single strip, the author applies methods similar to those used in his earlier studies of single airfoils [Philos. Trans. Roy. Soc. London. Ser. A. **247**, 131–162 (1954); MR 16, 414]. In the present application, the customary Kutta-Joukowski condition of smooth flow at trailing edges is applied and shed vortices are assumed to drift in the direction of the mean flow with the undisturbed stream speed. Integral formulas are obtained for lift, pitching moment, pressure distribution, and wake strength in terms of the upwash velocity distribution, which is identical at all airfoils at any time. These involve the associated Legendre functions of first order $Q^1_{p/r}$. These formulas are specialized for the case of upwash varying sinusoidally with time. They reduce to known results for single airfoils when the gap between airfoils tends toward infinity.

This work is related to an earlier study of unstaggered cascades of blades oscillating in phase [A. Mendelson and R. W. Carroll, NACA Tech. Note no. **3263** (1954); MR 16, 415], although the results are not in convenient form for comparison because different special functions were used. The present author's introductory remarks indicate that he proposes to apply his results to the blade-row interference problem studied by Kemp and Sears [J. Aero. Sci. **20**, 585–597, 612 (1953)]. It is to be noted, however, that this problem involves phase differences between upwash velocities at the various airfoils, in general. *W. R. Sears.*

Rennemann, Conrad, Jr. Minimum-drag bodies of revolution in a nonuniform supersonic flow field. NACA Tech. Note no. **3369**, 25 pp. (1955).

A formula is worked out for the drag of a slender body of revolution in a slightly nonuniform supersonic stream. The nonuniformity is expressed by a power-series expansion of its velocity potential about the axis of the slender body. The minimum-drag body for given length and volume is then sought. The result gives the cross-sectional area distribution in the form of the distribution for minimum drag in a uniform stream plus a correction term depending on the nonuniformity. This correction term seems to be very small, for practical cases, such as those in which drag reduction by mutual interference is employed. (It was previously shown by Friedman and Cohen [NACA Tech. Notes no. **3345** (1954)] that such drag reduction may be significant.)

W. R. Sears (Ithaca, N. Y.).

Broer, L. J. F. Pressure effects of relaxation and bulk viscosity in gas motion. Appl. Sci. Res. A. **5**, 55–64 (1954).

The author attempts to distinguish between the effects of relaxation mechanisms and those of viscosity and thermal conductivity. He calculates the Bernoulli integral for a gas in which bulk viscosity is far larger than shear viscosity. He then calculates the increase of entropy alone in a small tube inserted in a homogeneous flow, using a method of approximation due to Kantrowitz. His interpretation of the result is that "it is not correct to ascribe a very large coefficient of bulk viscosity to carbon dioxide on account of experiments on streaming." [However, all the mathematics is approximate and even so there are no detailed comparisons.] The paper closes with a corresponding treatment of one-dimensional flow. *C. Truesdell.*

Marchetti, Luigi. Caratteristiche aerodinamiche di particolari corpi di rivoluzione in moto in un'atmosfera molto rarefatta. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(**86**), 303–314 (1953).

The drag of a particular body of revolution in axisymmetrical free molecule flow with specular reflection at the boundary is determined as a function of Mach number. The body is that obtained by rotating an arc of a circle about a chord and cutting the resulting body off square at the rear by a plane at right angles to the said chord.

M. J. Lighthill (Manchester).

***Carrier, G. F., and Munk, W. H. On the diffusion of tides into permeable rock.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 89–96. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

Les auteurs étudient l'influence théorique de la marée sur la surface libre d'une nappe d'eau dans un milieu poreux en contact avec un océan. Cela conduit au problème linéaire aux limites suivant. Dans un secteur $0x$, $0t$, d'angle au sommet α, du plan vertical $0xy$ du mouvement ($0x$ horizontal représente la ligne libre moyenne dans le milieu poreux et $0t$ le fond de l'océan) on demande de construire $\varphi(x, y)$ complexe, solution de

1) $$\Delta\varphi - i\epsilon\varphi = 0 \quad (\epsilon > 0),$$

2) $$\varphi = 1 \quad \text{sur} \quad 0t,$$

3) $$\varphi_y + i\varphi = 0 \quad \text{sur} \quad 0x.$$

Le problème est traité dans les deux cas $\alpha = \pi$ et $\alpha = \frac{1}{2}\pi$ et il est résolu en utilisant la transformation de Fourier et les méthodes de Wiener-Hopf. L'équation de la ligne libre est une fonction périodique, de période égale à celle de la marée, l'amplitude étant de la forme $B(x)e^{i\mu(x)}$ où $B(x)$ est une fonction décroissante et $\mu(x)$ une fonction croissante. Les courbes représentatives sont construites pour certaines valeurs de ϵ (lié au coefficient de porosité). Ces résultats sont confrontés avec les observations effectuées par Cox sur les puits d'irrigation de Hawaii. L'accord est bon en ce qui concerne les amplitudes. Par contre, la longuer d'onde observvée est très inférieure à celle que donne cette théorie.

R. Gerber (Toulon).

Thompson, Philip D. A note on the integration of the vorticity equation for quasi-geostrophic flow. Tellus **6** (1954), 319–325 (1955).

The quasi-geostrophic equations for barotropic and certain simple types of baroclinic flow are transformed in such a manner that the isobaric height tendency is expressed in terms of first derivatives of the heights of isobaric surfaces. This shows that the solution of the quasi-geostrophic vorticity equation does in fact tend to average out the nonsystematic errors introduced by second and third order differences and so removes one of the most commonly raised objections against some recent methods of numerical weather prediction, namely that they require calculation of higher order derivatives of quantities which can only be measured

at widely separate points. The author treats the case of barotropic flow in detail and then shows a similar result is true for certain simple types of baroclinic flow. Finally, a graphical method for the calculation of the vorticity advection integral is given. *M. H. Rogers* (Urbana, Ill.).

Franz, Walter. Über die Greenschen Funktionen des Zylinders und der Kugel. Z. Naturf. **9a**, 705–716 (1954).

This paper is a continuation of an earlier work by the author and his co-worker K. Deppermann [Ann. Physik (6) **10**, 361–373 (1952); **14**, 253–264 (1954); MR **14**, 518; **16**, 97]. Consider the diffraction of a plane wave at a soft cylinder. From the Hankel-function series solution of the wave equation, a similar series may be obtained for the normal derivative of the wave function at the surface of the cylinder. For values of the radius large compared to the wavelength, this series is weakly convergent. By Watson's transformation, via an integral representation, this series is transformed into a series of residues, involving the zeros of the Hankel function considered as a function of the order. The transformed series is convergent on the shadow side, divergent on the illuminated side of the cylinder. [This seems to have been proved first in the analogous case of the sphere by E. Weibel in an unpublished Zürich dissertation, contrary to the belief of Sommerfeld that the residue series is convergent throughout (in fact, this is so for a source at finite distance from the cylinder). Weibel's results are indeed confirmed by the present author.] To overcome this difficulty, the author first splits off the exact expression for the geometrically reflected wave, so that the modified residue series turns out to be convergent on the illuminated side and the neighboring parts of the shadow region. The modified residue series appears identical with the author's earlier "creeping waves" derived from integral-equation formulations. This all is extended to the complete Green's functions of the cylinder and the sphere. A detailed numerical discussion is included. The earlier expressions obtained for the creeping waves are confirmed. The appendix contains additional material on the zeros of the Hankel function, Watson transformation versus integral equations, and asymptotics of Legendre functions.
C. J. Bouwkamp (Eindhoven).

Elasticity, Plasticity

***Swainger, Keith. Analysis of deformation. Vol. 2. Experiment and applied theory.** Chapman & Hall Ltd, London, 1954. xxxvi+365 pp. 70 s.

The criticisms given in the review of Vol. 1 [MR **16**, 307] apply equally to the present volume, which is devoted to solution of special problems and comparison of the results with experiment. Since sometimes a concrete example makes a previously ill-explained general theory clear, the reviewer attempted to follow Chapter II, "One-stress theoretical considerations," but altogether without success. Much effort has been put into this volume, which contains remarks on a great variety of experimental and theoretical researches.
C. Truesdell (Bloomington, Ind.).

Noll, Walter. On the continuity of the solid and fluid states. J. Rational Mech. Anal. **4**, 3–81 (1955).

The author is primarily concerned with materials, called hygrosteric, for which the constitutive equations specify the stress rate in terms of stress and velocity gradients. He begins by formulating an invariance requirement, called the principle of isotropy of space, which is intended to apply to all constitutive equations. Superimposing a rigid motion on a given motion induces certain transformations on all tensors appearing in the constitutive equations, the transformations being those which one would write down if one regarded the rigid motion as a coordinate transformation. His principle asserts that the form of the constitutive equations should be unaltered by any such transformation. The constitutive equations may involve variables referring to a particular reference position. A material is called isotropic (homogeneous) if its constitutive equations are unaltered when the reference configuration is replaced by any other obtainable from it by a rigid rotation (translation). He uses these requirements to show that, in constitutive equations involving stress, position coordinates, displacement gradients, and time derivatives of these quantities, the position, velocity and higher time derivatives of position cannot occur explicitly. For isotropic materials, Cotter and Rivlin [Brown Univ. Tech. Rep. no. **A11-113/12** (1954)] have obtained further restrictions using equivalent invariance requirements.

The rest of the paper is devoted to hygrosteric materials. After indicating what types of constitutive equations exhibit the required invariance, he distinguishes two types of hygrosteric materials, fluent bodies, which can remain at rest with time-independent stress only if the stress reduces to a hydrostatic pressure depending on the density, and resilient bodies, which can remain at rest with essentially arbitrary time-independent stress. He indicates that Reiner-Rivlin fluids may be regarded as limiting cases of fluent bodies, while elastic bodies are special types of resilient bodies. The relation between various types of ideal materials is graphically illustrated in a diagram on p. 47. His discussion (§16) of different assumptions which lead to equivalent results concerning the admissible forms of the strain energy in nonlinear elasticity is more inclusive than any other which this reviewer has seen. Also, initial stress is briefly discussed.

Stress relaxation and homogeneous stress of general hygrosteric bodies are discussed. Under very general conditions, if a fluent body remains at rest, the stress must relax to a hydrostatic pressure with the principal axes of stress remaining fixed in time. Existence and uniqueness of solutions for homogeneous stress for specified accelerationless flow is proved. For a special class of fluent bodies, solutions for rectilinear shearing motion, Poiseuille flow and Couette flow are obtained. One interesting fact is that, for certain ranges of values of material constants, steady Poiseuille flow is possible only for sufficiently small velocities. The author suggests that this might be relevant to the occurrence of turbulent flow.

There is not space here to discuss the solutions in detail. These are discussed adequately and rather thoroughly in the paper. *J. L. Ericksen* (Washington, D. C.).

Green, A. E. Finite elastic deformation of compressible isotropic bodies. Proc. Roy. Soc. London. Ser. A. **227**, 271–278 (1955).

In the general finite theory of elasticity for compressible isotropic bodies, the author sets up the problem of a circular cylindrical tube which is uniformly extended along its axis, uniformly inflated, and uniformly twisted about its axis. The corresponding solution for incompressible bodies is due to Rivlin [Philos. Trans. Roy. Soc. London. Ser. A. **242**,

173–195 (1949); MR 11, 627]. The author reduces the problem for the compressible case to a single ordinary differential equation of very complicated form. He does not discuss solution of this equation, but without solving it he is able to calculate the resultant twisting moment, obtaining a formula which in case the cylindrical boundaries are free of traction reduces to the same form as Rivlin's for the incompressible case [Proc. Cambridge Philos. Soc. **45**, 485–487 (1949); MR **10**, 650]. [There is no discussion of when, if ever, the cylindrical faces may be free of traction.]

Next the author sets up the problem of symmetrical expansion of a spherical shell. The solution for an incompressible material has been given previously by Green and Shield [Proc. Roy. Soc. London. Ser. A. **202**, 407–419 (1950); MR **12**, 218]. For the compressible case, the author obtains a non-linear integral equation for unstrained general radius ρ corresponding to the strained radius r. He does not discuss the solution of this equation. A corollary is the general condition for possible eversion of the shell.

C. Truesdell (Bloomington, Ind.).

Jindra, F. Die Hohlkugel bei einem nichtlinearen Elastizitätsgesetz. Ing.-Arch. **22**, 411–418 (1954).

This paper is a continuation of an earlier one [Ing.-Arch. **22**, 121–144 (1954)], and the same general comments [MR **16**, 88] apply also to this instalment, which treats radial stress in a hollow sphere [for the general solution of this problem, see the paper reviewed above]. The author concludes that his work shows that very small departures from Hooke's law can induce strong increases of stress concentration. *C. Truesdell* (Bloomington, Ind.).

Adkins, J. E., Green, A. E., and Nicholas, G. C. Two-dimensional theory of elasticity for finite deformations. Philos. Trans. Roy. Soc. London. Ser. A. **247**, 279–306 (1954).

The authors set up the equations for plane stress in general finite elasticity. The procedure is straightforward. Equations in terms of stress resultants are obtained in the classical manner by integrating the equations of equilibrium across the thickness of the plate. The stress-strain relations are specialized in a fashion asserted to be appropriate to "thin" plates even for finite deformations. There follows a formulation in complex variables with a perturbation series of the type now growing common in this subject. [The work is entirely formal, and no notice is taken of the conceptual difficulties inherent in "approximation" procedures for the finite theory.] *C. Truesdell* (Bloomington, Ind.).

***Signorini, A. Una espressiva applicazione delle proprietà di media dello stress comuni a tutti i sistemi continui.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 274–277. Academic Press Inc., New York, 1954. $9.00.

The author and his students have given much study to the mean values of the stress in continuous media. Various types of means can be evaluated from a knowledge of the loads alone, independently of such constitutive equations as the material may obey [A. Signorini, Ann. Scuola Norm. Sup. Pisa (2) **2**, 231–251 (1933), and later papers]. The reviewer notes that outside of Italy little attention seems to have been paid to these elegant and important results, by which properties of the stress system may be inferred even when the dynamical nature of the medium is not known. In the present paper the author briefly recalls some of his earlier theorems and points out that they can yield necessary conditions that the stress do not exceed an assigned bound. He derives such a condition, a quadratic inequality to be satisfied by certain mean values of the stress and certain geometrical properties of the body. For the case of the normal stress across a normal section of a prismatic body, the author gives an interpretation in terms of the bending moments about the principal axes of the section. *C. Truesdell* (Bloomington, Ind.).

Meixner, J. Thermodynamische Erweiterung der Nachwirkungstheorie. Z. Physik **139**, 30–43 (1954).

This contribution to the ever more popular linear theories of generalized continua comes from a thermodynamic background. The author writes σ_0 and ϵ^0 for the increments of temperature and specific entropy, σ_i and ϵ^i $(i=1, 2, \cdots, 6)$ for symmetric stress and (presumably linearized) strain. The theory he studies is of the Volterra type:

$$\sigma_i(t)=c_{jk}\epsilon^k(t)-\int_0^\infty C_{ik}(u)\epsilon^k(t-u)\,du \quad (i=0, 1, \cdots, 6).$$

The novelty here is the role of entropy (as in thermodynamics) instead of dilatation (as in the usual thermoelasticity) and in addition of thermic effects, with general linear interaction, to the usual Volterra theory. Beyond this, the author's results seem similar to those given in the large Italian literature [cf., e.g., D. Graffi, Univ. e Politec. Torino. Rend. Sem. Mat. **10**, 51–66 (1951); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 25–30 (1951); Atti Accad. Ligure **9**, 77–83 (1953); MR **13**, 800, 183; **15**, 182], and his methods, resting heavily on the Laplace transform, the same. His difference in view leads to emphasis on symmetry and definiteness of various matrices as consequences of now widely accepted thermodynamic principles. A particularly interesting conclusion appears in §7: in the isotropic case, the Laplace transforms of the stresses and strains satisfy equations for a linear visco-elastic material with frequency-dependent viscosities. All the work concerns the constitutive equations only. The field equations of continuous media are not mentioned. Thus in this paper there are no solutions of actual problems concerning flow or deformation. *C. Truesdell* (Bloomington, Ind.).

Bressan, Aldo. Sull'integrabilità del problema di de Saint Venant nei solidi cristallini. Rend. Sem. Mat. Univ. Padova **23**, 435–448 (1954).

The author considers St. Venant's problem for a general linearly elastic material. He shows that five of the six conditions of compatibility form by themselves a system whose solution is unique. Such a solution will then satisfy the sixth equation only in exceptional cases. Thus in general St. Venant's problem is insoluble. The author proceeds to characterize the special cases of elastic symmetry and/or form of cross-section for which St. Venant's hypotheses are compatible. *C. Truesdell* (Bloomington, Ind.).

Nowiński, Jerzy, and Turski, Stanisław. On the theory of nonhomogeneous isotropic bodies. Arch. Mech. Stos. **5**, 67–88 (1953). (Polish. Russian summary)

The classical theory of elasticity uses a model of a homogeneous body whose elastic properties (Young's modulus and Poisson's ratio) at every point are the same. If the elastic properties at a given point are the same in all directions but are different at different points then a body is said to be nonhomogeneous isotropic. Elastic properties at any

point of such a body are functions of coordinates. The authors assume that the Poisson's ratio remains constant and only Young's modulus is a continuous differentiable function of coordinates. The equilibrium stress equations and the continuity-of-displacements equations (Saint-Venant's compatibility equations) will remain for a non-homogeneous body the same as in the classical theory, they are true for any continuous medium whatever its properties, but the Beltrami-Michell compatibility equations containing the generalized Hooke's law would change into partial differential equations with variable coefficients. The authors illustrate the theory by two one-dimensional and two plane elasticity problems. The one-dimensional problems are as follows: (1) longitudinal vibrations of a beam whose Young's modulus is a linear function of one coordinate, $E=kx$; (2) torsion of a cylindrical bar whose shear modulus is symmetrical about the axis, $G=G(r)$. The plane-elasticity problems are as follows: (1) Lamé problem of a hollow thick-walled cylinder submitted to uniform pressure on inner and outer surfaces, whose shear modulus is a linear function of radius vector, $G=\alpha r$; (2) bending by an arbitrary load of a rectangular strip whose Young's modulus is an exponential function of one of the coordinates.

The reviewer noticed the following error: page 75, formula (2.16), should be $u(a, t)=0$, instead of $u(0, t)=0$.

T. Leser (Aberdeen, Md.).

Nowiński, Jerzy, and Turski, Stanisław. A study of states of stress in unhomogeneous bodies. Arch. Mech. Stos. 5, 397–414 (1953). (Polish. Russian and English summaries)

This is a continuation of the paper reviewed above. The authors develop further the theory of continuous nonhomogeneity and present a few more examples. They consider only prismatic or cylindrical beams with longitudinal elements parallel to the axis of the beam where all normal cross-sections are equal. The z-axis is parallel to the beam axis, the xy-plane passes through one of the normal cross-sections. The authors consider three kinds of nonhomogeneity: (1) transversal nonhomogeneity, Young's modulus, $E=E_{xy}(x, y)$; (2) longitudinal nonhomogeneity, $E=E_z(z)$; (3) transversal and longitudinal nonhomogeneity

$$E=E_{xy}(x, y)E_z(z).$$

They define a reduced centroid of a cross-section and a reduced moment of inertia. The considerations of strain energy show that variational principle can be easily applied. The authors give an example of Lamé's problem in the form known from the previous paper but solved by the application of Lagrange's variational principle. Transversal nonhomogeneity is illustrated further by problems of pure bending and torsion. Longitudinal nonhomogeneity is illustrated by a problem of buckling of a slender rod. It is shown from the bending of a plate with transversal nonhomogeneity that the Sophie Germain formula is applicable if the ordinary flexural rigidity is replaced by a so-called reduced flexural rigidity.

T. Leser (Aberdeen, Md.).

Marziani, Marziano. Sulla integrazione delle equazioni della elastodinamica. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 88, 55–67 (1954).

Consider an isotropic elastic body occupying the volume v bounded by a regular surface σ. It is required to find at a point P_1 of v and at the instant t the regular displacement $\mathbf{S}(P_1, t)$ which satisfies the equation

$$b^2 \operatorname{grad} \operatorname{div} \mathbf{S}-c^2 \operatorname{rot} \operatorname{rot} \mathbf{S}+\mathbf{F}=\partial^2\mathbf{S}/\partial t^2,$$

where b and c are positive constants, given the initial values of $\mathbf{S}$ and $\partial\mathbf{S}/\partial t$ and the external body and surface forces $\mathbf{F}(P, t)$, $\mathbf{R}(P, t)$ for $t>0$ and also the displacement $\mathbf{S}$ on the boundary σ.

By considering the Laplace transformation of the problem, the author is able to derive in a simple manner the classical results of O. Tedone [Mem. Accad. Sci. Torino (2) **47**, 181–258 (1897); Atti Accad. Sci. Torino **42**, 6–13, 516–521 (1907)] and A. E. H. Love [Proc. London Math. Soc. (2) **1**, 291–344 (1904)].

E. T. Copson.

Hu, Hai-Chang. On the equilibrium of a transversely isotropic elastic half space. Sci. Sinica 3, 463–479 (1954).

A general solution of the equations of linear elasticity for transversely isotropic materials, derived in a previous paper by the author [Acta Sci. Sinica **2**, 145–151 (1953); MR 15, 1004], is used to obtain solutions of six types of boundary-value problems for a half-space. These include solutions for specified surface tractions, displacements, normal load and tangential displacements, tangential load and normal displacements, a type of punch problem and bending of a plate by normal forces, one face of the plate being in contact with the half-space considered. The author indicates how, in the last two problems, one can obtain solutions from solutions for isotropic materials.

J. L. Ericksen.

Arf, C. Sur les frontières libres à tensions constantes d'un milieu élastique plan en équilibre. Rev. Fac. Sci. Univ. Istanbul. Sér. A. 19, 119–132 (1954). (Turkish summary)

The conditions of equilibrium of a plane elastic medium under no body forces and when $z=x+iy$ are known to give a solution for the stress components σ_x, σ_y and τ in the form

$$\sigma_x=\operatorname{Re}[2f(z)+\psi(z)-\bar{z}f'(z)],$$
$$\sigma_y=\operatorname{Re}[2f(z)-\psi(z)+\bar{z}f'(z)],$$
$$\tau=\operatorname{Im}[-\psi(z)+\bar{z}f'(z)],$$

where $f(z)$ and $\psi(z)$ are analytic functions. The author discusses in this paper the problem of determining the functions $f(z)$ and $\psi(z)$ defined in such plane domains which are such that the boundaries of the domains are free boundaries under constant tangential stress, i.e. such that

$$4 \operatorname{Re} f(z)=\sigma_x+\sigma_y=4\alpha,$$

where α is constant. The simple case of $f(z)$ being a constant over the whole plane is treated first and then the form of the free boundary in the z-plane can be defined in terms of an analytic function $\chi(\zeta')$ which takes real values on some parts of the real axis in the ζ'-plane. The author goes on to treat the cases where the free boundary is required to be a straight line, then a curve with a given radius of curvature, and finally finds the forms of solutions for the functions in the case when the free boundaries are two straight lines having a different constant tangential stress on each.

R. M. Morris (Cardiff).

Podstrigač, Ya. S. Action of a concentrated force on the boundary of a half-plane with a circular opening. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 217–219 (1954). (Ukrainian. Russian summary)

Föppl, L. Ein neues Auswerteverfahren der ebenen Spannungsoptik. Z. Angew. Math. Mech. 34, 454–459 (1954). (English, French and Russian summaries)

The author takes the net of lines of principal stress as the coordinate lines in the analysis of plane stress. The equations

of equilibrium are then formulated in terms of an analytic function, whose real part is the sum of the two principal stresses, and are integrated along an arbitrary path. The usefulness of the procedure stems from the fact that the integrand depends only upon the difference between the principal stresses and upon their directions, and hence is photoelastically determinate. *F. B. Hildebrand.*

Rostovcev, N. A. On the problem of torsion of an elastic half-space. Prikl. Mat. Meh. **19**, 55–60 (1955). (Russian)

The paper contains a solution of an axially-symmetric problem concerned with the determination of stresses and displacements in an elastic half-space twisted about the axis of symmetry by a rigid stamp connected with the half-space by friction or cohesion. *I. S. Sokolnikoff.*

Ševlyakov, Yu. A., and Zigel', F. S. Torsion of a hollow cylinder with an opening on the lateral surface. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 41–44 (1954). (Ukrainian. Russian summary)

Lakshmana Rao, S. K., and Sundara Raja Iyengar, K. T. Problems connected with the rhombus. I. Elastic torsion. J. Indian Inst. Sci. Sect. B. **36**, 159–171 (1954).

The authors study the torsion problem for a rhombus of side a and angles 2θ, $\pi-2\theta$. Using the Rayleigh-Ritz method with polynomials of degree 6 satisfying the boundary condition, and a method of Friedrichs with harmonic polynomials of degree 4, they obtain lower and upper bounds for the torsional rigidity D for arbitrary values of θ. For $2\theta=60°$, $0.09093<T<0.0951$, where $T=D/(\mu a^4)$. Comparison with a circle of the same area gives $T<0.1194$, and symmetrisation with respect to a perpendicular to a side gives $T<0.1051$. The bound $T>0.0433$ is obtained by regarding the rhombus as the union of two equilateral triangles. Finally, they triangulate the rhombus and use relaxation methods, obtaining an approximation $T=0.09121$. All the numerical results quoted above are for $2\theta=60°$. [The authors do not seem to be aware that the method of triangulation might have been used to obtain rigorous bounds; cf. J. L. Synge, Rend. Mat. e Appl. (5) **10**, 24–44 (1951); Proc. Symposia Appl. Math., vol. IV, McGraw-Hill, New York-Toronto-London, 1953, pp. 141–165; MR **16**, 257; **15**, 257.] *J. L. Synge.*

Narasimhamurthy, P. Torsion of multiply connected sections. J. Indian Inst. Sci. Sect. B. **36**, 150–158 (1954).

In the torsion problem for a hollow shaft we have to find a stress function ϕ to satisfy $\Delta\phi=-2$ with $\phi=0$ on the outer boundary and $\phi=C$ on the inner boundary B, the constant C being determined by the condition $\oint_B (\partial\phi/\partial n)ds=-2A$, where A is the area inside B. The author is concerned with the approximate calculation of C, the solution ϕ_1 for the solid shaft with the same external boundary being given. He uses the formula (*) $C^2=\int_0^{2\pi}\phi_1^2\,ds/2\pi$, where the integration is around B, the parameter s running from 0 to 2π. The formula (*) is deduced as an approximation from another formula (**) $\iint\phi_1\,dxdy=\iint\phi\,dxdy$, the integrals being taken through the material of the hollow shaft. Two numerical solutions are worked out with satisfactory results. [The argument is very difficult to follow. The formulae (*) and (**) are true for a ring bounded by concentric circles, and apparently this is the fact which gives the work some practical validity. The author takes the inner boundary circular, but the outer boundary is general, and the method seems to be without any proper mathematical justification. He appears to assert that (**) is true in general, which cannot be correct.] *J. L. Synge* (Dublin).

Craven, A. H. Torsion of cylinders with inclusions. Mathematika **1**, 96–103 (1954).

This paper deals with the problem of Saint-Venant torsion for cross-sections with solid or hollow inclusions, the elastic properties of which differ from those of the surrounding material. The method used is an extension of the complex-variable treatment employed by A. C. Stevenson [Phil. Mag. (7) **39**, 297–303 (1948); MR **10**, 272] in conjunction with homogeneous hollow cross-sections. Let C_i with radius r_i ($i=1, 2, 3$; $r_1<r_2<r_3$) be concentric circles about the origin of the ζ-plane; let Γ_i be the image in the z-plane of C_i with respect to a mapping $z=f(\zeta)$ which admits a Laurent expansion about $\zeta=0$. Series solutions are obtained for the case in which the two annuli bounded by Γ_1, Γ_2 and Γ_2, Γ_3 are occupied by different elastic materials (hollow inclusion) and for the case in which the region interior to Γ_1 and the annulus bounded by Γ_1, Γ_2 are occupied by different materials (solid inclusion). The mapping $f(\zeta)=c(1+\lambda\zeta^n)$, for suitable choices of c, λ, n, yields a nearly circular outer annulus and an inner annulus whose inner boundary is approximately polygonal; it also yields approximately polygonal sections with nearly circular annular inclusions. The torsional rigidities are given in detail for several specific examples. *E. Sternberg* (Chicago, Ill.).

Huber, Alfred. The elastic sphere under concentrated torques. Quart. Appl. Math. **13**, 98–102 (1955).

This paper contains a solution in closed form, and in terms of elementary functions, of the problem of pure torsion of an elastic sphere subjected to two equal and opposite concentrated couples applied at the endpoints of a diameter. The solution is exact within the classical theory of elasticity. The problem is formulated in terms of the stress function for torsion of circular shafts of variable cross-section. The results are reached with remarkable economy on the basis of generalized potential-theoretic considerations due to A. Weinstein [Trans. Amer. Math. Soc. **63**, 342–354 (1948); Bull. Amer. Math. Soc. **59**, 20–38 (1953); MR **9**, 584; **14**, 749]. It is shown that the boundary-value problem characterizing the stress-function Φ has a unique solution provided Φ is bounded in the semi-meridional region under consideration. It is further shown that the solution thus obtained coincides with the limit of a sequence of solutions appropriate to homogeneous distributions of shearing tractions over two spherical caps as the caps are contracted toward the points of application of the concentrated couples while the respective stress resultants tend to the prescribed couples. To demonstrate the need for his concern with the uniqueness of the solution, the author exhibits an elementary stress function Φ^* which meets all requirements imposed on Φ except for boundedness: This stress function, in conjunction with Φ, may be used to generate an infinity of "pseudo-solutions" all of which conform to the traditional formulation of the problem but fail to coincide with the limit solution described above. *E. Sternberg.*

Mossakovskiĭ, V. I. Pressure on an elastic half-space of a die which is almost circular in planform. Prikl. Mat. Meh. **18**, 675–680 (1954). (Russian)

The area of contact of the die with the half-space is assumed to have a centre of symmetry and the contact is assumed to be smooth. The problem in which the area of

contact is replaced by its largest inscribed circle is solved by the use of certain contour integral representations given by the author in another paper [Prikl. Mat. Meh. 18, 187–196 (1954); MR 16, 539] and this solution is used to obtain an approximate solution of the given problem. The method is applied to an elliptic area of contact.
L. M. Milne-Thomson (Greenwich).

Kočetkov, A. M. The stressed state of a wedge under the action of hydrostatic pressure. Inžen. Sb. 15, 177–180 (1953). (Russian)

Ambarcumyan, S. A. On the computation of long shells of double curvature. Akad. Nauk Armyan. SSR. Izvestiya. Fiz.-Mat. Estest. Nauk 6, no. 5–6, 65–68 (1953). (Russian. Armenian summary)

The author considers a very sloping shell (a sloping shell is an open shell of small curvatures) which is made up of orthotropic layers. Solutions for such a shell are obtained from a system of two differential equations given by the author in his previous (unavailable) publications. In case of a long shell (the exact definition of a long shell must be also in the author's previous publications) the system of differential equations simplifies considerably and reduces to the one given by V. Z. Vlasov [General theory of shells, Gostehizdat, Moscow-Leningrad, 1949; MR 11, 627]. The author makes one more simplifying assumption that the Poisson ratios are zero and solves the system for a long cylindrical sloping shell curved in the longitudinal direction. For a homogeneous isotropic shell the solution reduces to the one given by V. V. Novožilov [Theory of thin shells, Gostehizdat, Moscow-Leningrad, 1951] and the author concludes that Novožilov's theory and Vlasov's theory are coincident. *T. Leser* (Aberdeen, Md.).

Naghdi, P. M., and De Silva, C. Nevin. On the deformation of elastic shells of revolution. Quart. Appl. Math. 12, 369–374 (1955).

The authors deal with a formulation, due to E. Reissner [H. Reissner Anniversary Volume, Edwards, Ann Arbor, Mich., 1949, pp. 231–247; MR 11, 69], of the theory of small deformations of elastic shells of revolution. It is shown that the two basic second-order differential equations, upon the solution of which the analysis is made to depend, can be combined into a single complex equation of second order when a certain relationship exists between the geometry of the middle surface and the thickness variation. Conical and toroidal shells of uniform thickness are admitted as special cases. *F. B. Hildebrand.*

Cywińska, Zofia, and Mossakowski, Jerzy. The influence surfaces of an orthotropic semi-infinite strip. Arch. Mech. Stos. 6, 33–64 (1954). (Polish. Russian and English summaries)

Nearly all existing solutions of an orthotropic semi-infinite strip are in terms of slowly convergent trigonometric series. In this paper the authors make a successful effort to find solutions in a closed form summing up the series. The considered strip is loaded by a concentrated force or by a moment. The authors are interested mainly in having second partial derivatives of the deflection in a closed form because this is only needed to write down moments and shearing forces. The authors found also that solutions of all the examples contain various combinations of two basic functions. They propose a close study and tabulation of these functions.
T. Leser (Aberdeen, Md.).

Nowiński, Jerzy. The deflection of a circular membrane supported on an elastic ring. Arch. Mech. Stos. 5, 295–307 (1953). (Polish. English summary)

A membrane described in the title is subjected to a constant normal distributed load. Bottoms of certain cylindrical containers can be regarded as such membranes. Deflections and stress function for membranes are controlled by a system of two partial differential equations known as equations of Kármán. For a circular membrane this system was solved by H. Hencky [Z. Math. Phys. 63, 311–317 (1914)] in rapidly convergent series whose coefficients have to be determined from boundary conditions. The author finds the coefficients of the first four terms from the condition that on the boundary the circumferential strain of the plate must equal that of the ring. He discusses his solution with the aid of graphs. The loss of stability is excluded.
T. Leser (Aberdeen, Md.).

Kączkowski, Zbigniew. Representation of the functions of deflection of an infinite strip in closed forms. Arch. Mech. Stos. 5 (1953), 589–628 (1954). (Polish. Russian and English summaries)

A solution of a problem of an infinite strip is usually given in terms of trigonometric series which converge rather slowly. For an infinite plate though, especially when loads are circularly symmetric, a solution in a closed form is easily obtained due to simple boundary conditions. In order to solve an infinite strip in a closed form the author makes it a special case of an infinite plate. Closed forms which the author obtains are sums of series containing logarithmic expressions. The first article of this paper deals with the theory of summation of this kind of series. The author presents several examples for various loads and boundary conditions. In all of them solutions contain combinations of four basic functions in a closed form. It is very probable that many of the infinite-strip problems can be solved in terms of these four functions, which suggests desirability of tabulating them. *T. Leser* (Aberdeen, Md.).

Uzdalev, A. I. Bending of an anisotropic two-layered cylinder by a transverse force. Inžen. Sb. 15, 35–42 (1953). (Russian)

The author gives a solution of the problem stated in the title for the special case of a circular cylinder with curvilinear (cylindrical) anisotropy and when the Poisson ratios of material in the radial and tangential directions, in both layers, are identical. *I. S. Sokolnikoff.*

Nowacki, Witold, and Mossakowski, Jerzy. The influence surfaces of plates representing annular sectors. Arch. Mech. Stos. 5, 237–272 (1953). (Polish. English summary)

The authors investigate the bending of plates in the shape of a circular sector loaded by concentrated forces and freely supported on the edges. They solve first the problem for an infinite wedge and use this solution for an annular sector. The particular cases of shapes and loads when the deflections, moments and shearing forces can be represented in a closed form are discussed in detail. The problem of an infinite strip is solved as a limiting case of a circular sector. In this way the solution constitutes a generalisation of results obtained by other authors for infinite and semi-infinite strips. *T. Leser* (Aberdeen, Md.).

Paria, Gunadhar. Stress distribution in thin aeolotropic plates. I. Bull. Calcutta Math. Soc. 46, 103–107 (1954).

This is a short paper in which it is shown that by the aid of the Fourier transform of the stress function, the fundamental equation satisfied by this stress function for a thin aeolotropic plate under generalised plane stress can be reduced to an ordinary differential equation. The solution of this differential equation consistent with the boundary conditions leads immediately to the determination of the stresses. The method is similar to that adopted by Sneddon [Fourier transforms, McGraw-Hill, New York, 1951, p. 442; MR 13, 29] for the solution of problems of isotropic plates. It is proposed to consider two specific problems in detail in a subsequent paper. *R. M. Morris* (Cardiff).

***Cesari, L., Conforto, F., e Minelli, C. Travi continue inflesse e sollecitate assialmente.** Consiglio Naz. Ricerche. Publ. Ist. Appl. Calcolo, no. 91. Roma, 1941. viii+244 pp. 2800 Lire (order from Rosenberg & Sellier, Via Andrea Doria 14, Torino).

This is a compilation of formulas and tables of coefficients relevant to a generalization of the Clapeyron three-moment equation to the case of a continuous beam with piecewise linear variation of bending stiffness, supported at a finite number of points and subject to a uniform transverse loading and to an axial thrust. *F. B. Hildebrand.*

Baldacci, Riccardo F. Sulle equazioni generali di stabilità del moto di un corpo elastico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 88, 38–45 (1954).

A discussion is given of the general problem of the dynamic stability of an elastic body under applied force. The governing differential equations and boundary conditions are established by Hamilton's principle. Applications of the analysis are not discussed. *H. G. Hopkins.*

Baldacci, Riccardo F. Sul problema linearizzato di stabilità dinamica per un corpo elastico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 88, 199–203 (1954).

The results of the analysis of the paper reviewed above are re-presented. The governing equations are simplified through the assumption that the space variables are separable from the time variable. The special form of these equations appropriate to beams and plates is stated without further analysis. It is notable that Hill's equation is of fundamental importance in the solution of problems.
H. G. Hopkins (Fort Halstead).

Weidenhammer, F. Der eingespannte, achsial pulsierend belastete Stab als Stabilitätsproblem. Ing.-Arch. 19, 162–191 (1951).

A bar is clamped at both ends and subjected to a harmonically oscillating axial force. Lateral vibrations result in some cases. The bulk of this paper is devoted to the question of whether these vibrations die down or increase, possibly catastrophically.

The problem is formulated as an extremum problem in the calculus of variations, beginning with Hamilton's Principle. A significant feature of this development is the fact that the author is able to introduce linear velocity damping and still reduce the problem to a variational principle by means of a transformation of the dependent variable involving a multiplicative exponential damping factor. This extremem problem is solved approximately using a modification of the Rayleigh-Ritz ideas. The approximating functions contain time-dependent factors satisfying a system of equations which reduce by means of a normal coordinate transformation to equations of Mathieu's type. These are investigated for stability in considerable detail, using Floquet and Hill theory as a starting point. *E. Pinney.*

Grammel, G. Zur Stabilität erzwungener Schwingungen elastischer Körper mit geschwindigkeitsproportionaler Dämpfung. Ing.-Arch. 20, 170–183 (1952).

The author makes use of the device used by Weidenhammer (cf. preceding review) to investigate the stability of elastic vibrations using a variation technique even though linear velocity damping is present. He formulates the problem in general terms, using the nonlinear expression for the strain tensor in terms of the displacement derivatives, but using a linear stress-strain relation. First, neglecting friction, he begins with Hamilton's Principle, finally obtaining a partial differential equation system for the deviation of the displacement from an assumed vibration mode. To this he adds a linear velocity damping term. Then the displacement deviation is expressed as a new variable multiplied by an exponential damping factor which is such that the differential equation in the new variable is derivable from a variation principle.

The variation problem is solved by a modification of the Rayleigh-Ritz method. Time-dependent terms are made to depend on a system of Mathieu-type equations. This is treated in a theory involving expansion in terms of a small parameter which measures the amplitude of the oscillating term in the Mathieu equations. When this parameter is zero the system reduces to a constant coefficient system. The theory is applied to the vibrations of a rectangular bar, pin-jointed at the two ends. *E. Pinney.*

Naleszkiewicz, Jarosław. The quantization of the phenomena of elastic unstability. I. Arch. Mech. Stos. 6, 3–32 (1954). (Polish. Russian and English summaries)

In this paper "quantization" has no connection with Planck's quantum of energy. Different states of elastic equilibrium for the same external load occur at energy levels which differ by constant quantities. These constant energy differences the author calls quanta. He introduces a very simple model of an elastic system requiring only elementary mathematical tools. He investigates for his model various positions of stable equilibrium and the loss of stability. He shows the existence of different states of equilibrium for the same loads. He studies the potential energy of the system in order to explain why in practice there is only one position of stable equilibrium, the others are difficult to obtain, although mathematically they are also stable. The position of equilibrium occurring in practice corresponds not only to a local minimum but also to the least of all energy levels. This state of equilibrium is called absolutely stable. To raise the system from one state of equilibrium to the next one it is necessary to introduce an additional quantum of energy. *T. Leser* (Aberdeen, Md.).

Naleszkiewicz, Jarosław. The quantization of the phenomena of elastic unstability. II. Arch. Mech. Stos. 6, 261–290 (1954). (Polish. Russian and English summaries)

This paper is a continuation of the paper reviewed above. In the first paper the author studied on a simple model of an elastic system various states of equilibrium at different energy levels. In this paper the author studies a more com-

plicated model of a bar with uniform cross-section loaded by an axial force, a torque and a bending moment. The system of differential equations controlling such a model was derived by A. Grzedzielski and J. Nowiński [Calculations of three-bar pyramids, Sprawozd. Inst. Techn. Lotn. 4/26 (1938)]. The author introduces certain simplifying assumptions and solves the system, obtaining two expressions for deformations. For a beam in a critical state these expressions represent two different criteria for stability which depend on the torque. In the absence of torque they have different forms than those containing torque. The former case corresponds to a lower energy level the latter to a higher one. The example shows that in systems not as simple as the one discussed in the first paper the question of energy levels may become quite intricate. *T. Leser.*

Baldacci, Riccardo F. Sulla instabilità dinamica della lastra. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 88, 21–37 (1954).

A discussion is given of the dynamic stability of plates. The problem is first simplified in the usual way to one in two dimensions. The governing differential equations and boundary conditions are then established by Hamilton's principle. The analysis is illustrated mainly by application to a simply-supported rectangular plate under oscillatory forces applied in the middle surface and perpendicular to the edges. The effect of linear damping is also discussed and a numerical example is given. *H. G. Hopkins.*

Horne, M. R. The flexural-torsional buckling of members of symmetrical I-section under combined thrust and unequal terminal moments. Quart. J. Mech. Appl. Math. 7, 410–426 (1954).

This paper presents an iterative method for the approximate determination of the elastic buckling load of a symmetrical I-section member subjected to an axial thrust, applied through the centroid, and to unequal terminal bending moments, acting in the plane of the web, when account is taken of the reduction of torsional rigidity due to the axial load and of the effect of differential bending of the flanges. Simple conservative formulas are also deduced, and numerical results are presented in special cases.
F. B. Hildebrand (Cambridge, Mass.).

Neuber, H. Gesetzmässigkeiten von Torsionsschwingungszahlen und Aufstellung einfacher Grenzwertformeln. Ing.-Arch. 22, 258–267 (1954).

The free torsional vibrations of a system of n rigid disks, connected by $n-1$ massless elastic torsion bars, are considered here for the purpose of obtaining upper and lower bounds to the frequencies of oscillation. The equations of elasticity and motion are combined for simple harmonic vibrations into a single difference equation at each disk in terms of the torsional moments; these difference equations then are used to obtain the frequency equation in determinental and in polynomial form. It is shown that a lower bound on the fundamental frequency and an upper bound on the highest frequency can be obtained using the coefficients of the polynomial equations for the frequency squared and for the reciprocal of the frequency squared. An inverted set of bounds can be obtained by considering frequencies of simplified systems derived from the original by changes of stiffness and inertia. Having obtained good approximations for the two extreme frequencies, it is stated that intermediate frequencies may also be bounded.
W. Nachbar (Seattle, Wash.).

Nazarov, A. G. On the dissipation of energy in elastic vibrations. Akad. Nauk Armyan. SSR. Dokl. 16, 77–86 (1953). (Russian. Armenian summary)

Let the restoring force of a vibrating system with one degree of freedom be $Ky(t)$, where $y(t)$ is the displacement at time t. If $R(t)$ is the force due to internal friction, the author writes $R(t)=K[y(t+\tau)-y(t)]$, where it is assumed that $R(t)$ is sufficiently small compared with $Ky(t)$. Concerning τ two hypotheses are made: (I) that τ is constant; (II) that τ is inversely proportional to the frequency. The application of (I) leads to results violently in disagreement with observation, whereas the application of (II) shows good agreement. The idea is then developed in relation to a "monochromatic" elastic system by introducing two time intervals τ and τ_1, corresponding to normal and tangential stress, obeying (II), and leading to linear equations.
L. M. Milne-Thomson (Greenwich).

Nazarov, A. G. A method of computing the dissipation of energy in elastic vibrations. Akad. Nauk Armyan. SSR. Dokl. 16, 137–140 (1953). (Russian. Armenian summary)

The author shows how energy methods combined with calculations for the undamped system and experimental observations on the actual system can be used to set up sufficient equations to find approximately the law of dissipation. *L. M. Milne-Thomson* (Greenwich).

Bolotin, V. V. On bending oscillations of shafts whose sections have nonidentical principal flexural rigidities. Inžen. Sb. 19, 37–54 (1954). (Russian)

The paper concerns a shaft whose cross-section has unequal moments of inertia I_1, I_2 $(I_1<I_2)$ and therefore what the author calls a "coefficient of anisotropy"

$$\mu=\tfrac{1}{2}(I_2-I_1)/(I_2+I_1).$$

The shaft rotates with angular velocity θ about the horizontal line joining the end supports, and with the simplifications of the paper the sag of the mass-centre obeys the equation

$$f''+2\epsilon f'+\omega^2(1-2\mu\cos 2\theta t)+\psi(f,f',f'')=0,$$

where $\psi(f,f',f'')$ is of the form

$$\gamma f^3+2\epsilon f^2f'+2\kappa f[ff''+(f')^2],$$

γ being either a constant or containing a term proportional to $1-2\mu\cos 2\theta t$. By considering various cases including resonance $(\theta=\omega)$, the author is able to obtain values and exhibit graphs which throw light on the paradoxes which arise when the equation is linearized.
L. M. Milne-Thomson (Greenwich).

Nowacki, Witold. Vibrations and buckling of rectangular plates simply supported on the periphery and at several points inside. Arch. Mech. Stos. 5, 437–454 (1953). (Polish. Russian and English summaries)

The object of this investigation is to find the influence of a point support on a plate described in the title, which is compressed by loads acting on the opposite two edges. Such formulation of the problem led to problems of buckling and free vibrations in the absence of compressive loads. The author begins by solving a plate loaded normally by a vibratory concentrated load of frequency ω. He finds the deflection under the load, sets it equal to zero, and solves the resulting equation for the frequency ω, obtaining rotso $\omega_1,\omega_2,\ldots$. These roots represent natural vibration fre-

quencies of the plate supported at the point where the load acted previously. When the compressive loads increase, the natural frequency decreases and the load for which the frequency approaches zero is the critical load. In the next paragraph the author investigates the influence of the location of point support and of the dimension ratio of the plate on the critical load. He investigates also two or more point supports and one linear support. In the third paragraph the author analyzes free vibrations of a plate with one or more additional supports. *T. Leser* (Aberdeen, Md.).

Hvingiya, M. V. Small characteristic oscillations of a conical spring with constant pitch. Inžen. Sb. **16**, 73–80 (1953). (Russian)

The projection of a conical spring with constant pitch is a spiral of Archimedes which simplifies many computations. The spring is loaded with a constant force P. The oscillations are assumed to be small, that is, the neighbouring turns of the spring do not touch each other when the vibrating spring is compressed. Small oscillations of such a spring are controlled by the partial differential equation of a beam with variable rigidity. The author solves this differential equation in the usual way by separation of variables and easily obtains the general solution in terms of Bessel functions, but determination of the four constants of integration and the constant of separation (this last constant represents the frequency of vibrations) was quite difficult. These constants were, of course, found from boundary conditions. Conditions on the boundaries $x=0$, $x=H$ lead to transcendental characteristic equations. The author gives several tables of roots of these equations claiming that the error is much less than 5 per cent. The formula for the frequency is then as accurate as the characteristic number which it contains.

The author compares his exact formula with approximate formulas found by other investigators and shows that the approximate formulas give considerable errors for outside loads which are small as compared with spring's own weight. *T. Leser* (Lexington, Ky.).

Tveritin, O. M. Mathematical consideration of the problem of longitudinal impact on an elastic-viscous bar with free ends. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 307–312 (1953). (Ukrainian. Russian summary)

An elastic-viscous bar of constant cross-section is situated horizontally and its ends are free. At an initial moment the bar, at rest up to then, receives on its right end an impact from a weight which thereafter remains in contact with the bar. A solution of the problem is constructed by the author by the classical method of Fourier. In addition there is carried out an investigation of the solution and its partial derivatives of first and second orders and also of $\partial^3u/\partial x^2\partial t$, especially in relation to the behavior of the solution and the indicated derivatives as $t\rightarrow 0$, and also (for $\partial^2u/\partial x^2$ and $\partial^3u/\partial x^2\partial t$) for $x=l$ (the right-hand end). There is also treated the behavior of the function and its partial derivatives as $t\rightarrow\infty$. Finally there is also treated, under certain definite conditions, the question of uniqueness of the solution. *Author's summary.*

Irmay, S. Dynamic behaviour of linear rheological bodies under periodic stresses. Quart. J. Mech. Appl. Math. **7**, 399–409 (1954).

The basis of this investigation is the one-dimensional equation (1) $a_1p+a_2\dot{p}=a_3\epsilon+a_4\dot{\epsilon}$ relating the stress p, the strain ϵ, and their time derivatives. Various cases of applied periodical and transient stresses are considered and the corresponding strains are determined. Inertia effects are neglected. The physical significance of the constant coefficients of (1) is discussed in detail, and a classification is given accordingly. *W. Noll* (Berlin).

Swann, W. F. G. Shear modulus and viscosity relations in plastic materials. J. Franklin Inst. **259**, 11–16 (1955).

This paper presents in detail the author's reasoning leading to the results asserted in a previous note [J. Appl. Phys. **25**, 1108–1109 (1954); MR **16**, 197]. *C. Truesdell.*

Franciosi, Vincenzo. Il carico di punta critico in regime elastoplastico. Rend. Accad. Sci. Fis. Mat. Napoli (4) **20** (1953), 177–185 (1954).

Analysis is given of the plastic stability of a simply-supported ideal bar under pure compression load. The general results are in agreement with those predicted by F. R. Shanley [J. Aero. Sci. **14**, 261–267 (1947)] for a simplified model that simulates the behaviour of the bar. In particular, instability can first occur at the tangent-modulus load rather than at the reduced-modulus load.

The reviewer has these comments. In practice the above is the situation. This is because inevitable imperfections such as those of geometry and loading position, and also those of a dynamical nature due to the process of loading, are factors all tending to act in favour of the detection of instability at the least possible critical load. The same result applies to the plastic stability of more complicated structures. It is important to recognize that care is required in formulating the stability criterion for systems exhibiting irreversible behaviour. This fact was stressed by Th. von Kármán [ibid. **14**, 267–268 (1947)] in commenting on the paper by F. R. Shanley (loc. cit.). *H. G. Hopkins.*

Shield, R. T. The plastic indentation of a layer by a flat punch. Quart. Appl. Math. **13**, 27–46 (1955).

A perfectly smooth, rigid punch whose cross section is a square of side $2a$ is pressed against an infinite layer of perfectly rigid-plastic material of thickness h. The layer in turn rests upon a perfectly rough base. The problem is to determine the indentation pressure p at which the punch will just penetrate the rigid-plastic material. For the corresponding plane-strain problem Wang and Lee [Report AC-47 to Armstrong Cork Co., Brown University, 1953] found the exact indentation pressure p, but for the three-dimensional problem limit analysis theorems are used to obtain upper and lower bounds on p.

The case $h=\infty$ was previously treated by Shield and Drucker [J. Appl. Mech. **20**, 453–460 (1953)] and the same results are valid provided $a/h<1.79$. The present paper presents five velocity fields which give upper bounds for various ranges of a/h and three stress fields for lower bounds. As would be expected both bounds increase as a/h becomes larger. For all values considered the two bounds differ by less than 20% of their mean value.

The circular punch is also treated more briefly and it is pointed out that the lower bounds may easily be modified to apply to any convex punch. *P. G. Hodge, Jr.*

Mayers, J., and Budiansky, Bernard. Analysis of behavior of simply supported flat plates compressed beyond the buckling load into the plastic range. NACA Tech. Note no. 3368, 44 pp. (1955).

The plate chosen for study is a simply supported square sandwich plate which buckles in the elastic range. All edges are assumed constrained to remain straight. A large deflection post-buckling analysis is based upon the variational principle for deformation theory of plasticity which is the equivalent of the minimum potential energy theorem of elasticity. The minimum is not at all sharp so that computations were performed on the SEAC using a modified steepest descent procedure but in some cases convergence was still unsatisfactory. Coefficients in all of a double trigonometric expansion of the components of displacement were varied. As explained by the authors, the total difference in energy between the first cycle of iteration based on elastic displacement coefficients and the last cycle was extremely small, about 1%. The accurate determination of stress requires the lengthy minimization. *D. C. Drucker.*

Čobanyan, K. S. Stability of the plane form of bending beyond the elastic limit for an arbitrary law of hardening. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki. 6, no. 4, 1–20 (1953). (Russian. Armenian summary)

The critical length of a beam rendered partly plastic due to bending in a plane of symmetry is computed from the current torsional and flexural rigidities of the yielded section. The stress-strain law used is the small elastic-plastic deformation (total strain) form of Ilyušin [Plasticity, part I, OGIZ, Moscow-Leningrad, 1948; MR **12**, 373] and the velocity distribution is assumed to be the same as that of the wholly elastic beam. Numerical values are given for the rectangular beam and for an I beam. The author seems unaware of the work of Neal [Philos. Trans. Roy. Soc. London. Ser. A. **242**, 197–242 (1950)] who, incidentally, uses a more precise expression for the critical length, based on H. Reissner, S.-B. Berlin. Math. Ges. **3**, 53–56 (1904).
R. Haythornthwaite (Providence, R. I.).

MATHEMATICAL PHYSICS

Poincelot, Paul. Remarques sur la notion de vitesse de groupe; applications diverses. J. Math. Pures Appl. (9) **33**, 329–364 (1954).

This paper gives the detail of the author's method of treating the propagation of waves in a dispersive medium [C. R. Acad. Sci. Paris **234**, 599–602 (1952); **238**, 1289–1291 (1954); MR **13**, 599; **15**, 913]. A permanent regime for frequency ω is represented by $\exp i[\omega t-\phi(\omega)]$, where $\phi(\omega)=\omega n(\omega)x/c$, n = refractive index, c = light velocity in vacuo. In the domain $x>0$, a disturbance $f(x, t)$, for the boundary conditions $f(0, t)=0$ for $t<0$, $=\exp i\Omega t$ for $t>0$, is expressed as

$$f(x,t)=\frac{1}{2\pi i}\int_{C_1}(\omega-\Omega)^{-1}\exp i[\omega t-\phi(\omega)]\,d\omega,$$

C_1 being an infinite line parallel to, and below the real axis. This may be written $f=Fg$, where F is the steady regime for Ω, and

$$g(x,t)=\frac{1}{2\pi i}\int_{C_1}z^{-1}\exp i[zt-\phi(\Omega+z)+\phi(\Omega)]\,dz.$$

To link this with elementary ideas about group velocity, the author examines the error produced by replacing $\phi(\Omega+z)-\phi(\Omega)$ by $z\phi'(\Omega)$, a replacement which makes $g(x, t)$ zero for $t<t_g$ and unity for $t>t_g$, t_g being the group time ($=x/g$, where g is the group velocity). The rest of the paper is devoted to the case $n=(1-a^2/\omega^2)^{1/2}$, which applies to the ionosphere, to de Broglie waves, and to wave guides. The integral $f(x, t)$ is expressed in terms of Bessel functions, and, after examining asymptotic formulae in numerical instances, the author concludes that the wave front is propagated with velocity c and arrives with its initial amplitude, the signal, for some thousands of periods, taking the form

$$S(t)=(2/\pi a)^{1/2}(t^2-\tau^2)^{-1/4}\cos[a(t^2-\tau^2)^{1/2}-\pi/4],$$

where $\tau=x/c$; this represents the precursors or forerunners. At the group time $t_g=\tau(1-a^2\Omega^{-2})^{-1/2}$ the permanent regime appears suddenly. Reflection at a surface of discontinuity is discussed. In a terminal note the author acknowledges the priority of A. Rubinowicz [Acta Phys. Polon. **10**, 79–86 (1950); MR **15**, 919], stating that the expositions differ but the procedures agree [cf. MR **16**, 201 for references to a later paper by Rubinowicz and to papers by J. D. Pearson and M. Cerrillo]. *J. L. Synge* (Dublin).

Optics, Electromagnetic Theory

Vergeim, B. A., and Ostroumov, G. A. On the manifestation of optical inhomogeneities. Prikl. Mat. Meh. **19**, 109–112 (1955). (Russian)

Let $f(x, y, z)$ be the index of refraction within the rectangular parallelepiped $0\leq x\leq X$, $0\leq y\leq Y$, $0\leq z\leq Z$. Insert the parallelpiped in monochromatic beams parallel to each edge. Let the corresponding optical path lengths be

$$\Phi_1(y,z)=\int_0^X f\,dx,\quad \Phi_2(z,x)=\int_0^Y f\,dy,\quad \Phi_3(x,y)=\int_0^Z f\,dz.$$

Unless f is plane or axisymmetric this system does not have a unique solution. Now approximate f by a multinomial $P(x, y, z)$ of degree N chosen to minimize

$$\int_0^X\int_0^Y\int_0^Z (f-P)^2dxdydz.$$

The author shows that if Φ_1, Φ_2, Φ_3 are given, P is uniquely determined for $N=2$. If $\int_0^X xf\,dx$ is known, as in the symmetrical case $f(x, y, z)=f(X-x, y, z)$, P is unique for $N=3$; if all three first moments are known, for $N=5$. Similarly, if Φ_1, Φ_2, Φ_3 are multinomials of degree $N=1$ or 2, the solution for f as a multinomial of the same degree is unique. *J. H. Giese* (Havre de Grace, Md.).

Rubinowicz, A. Die Rolle der Beugungswelle in den Fraunhoferschen Beugungserscheinungen. Acta Phys. Polon. **13**, 3–13 (1954). (Russian summary)

The paper is, in a sense, a continuation of earlier work by the same author [Ann. Physik (4) **53**, 257–278 (1917); **73**, 339–364 (1924)]. The total field behind an aperture can be written as a sum of the geometric-optics field and a "diffraction wave" originating at the edge of the aperture, all in the scheme of Kirchhoff's theory. The present paper studies phenomena of discontinuities along the boundary of the geometric-optics field when the primary source goes to

infinity (plane-wave excitation) and the point of observation lies in the Fraunhofer zone. Reference is made to a transformation of M. Laue [S.-B. Preuss. Akad. Wiss. **1936**, 89–91]. *C. J. Bouwkamp* (Eindhoven).

Williams, W. E. Diffraction by two parallel planes of finite length. Proc. Cambridge Philos. Soc. **50**, 309–318 (1954).

A plane wave is incident upon a pair of parallel planes of finite length and "infinite" width. The electromagnetic diffraction problem (for the magnetic vector parallel to the edge of the plates) is formulated as a pair of integral equations and solved approximately for the case that the length of the planes is large compared to the wave-length. The resonance of the system is studied and an expression is found for the resonant lengths. *A. E. Heins* (Pittsburgh, Pa.).

Erdélyi, A., and Papas, C. H. On diffraction by a strip. Proc. Nat. Acad. Sci. U. S. A. **40**, 128–132 (1954).

The authors consider the diffraction of a plane wave by a perfectly conducting strip in the case of normal incidence with the electric field parallel to the edges of the strip. The electric field is expressed as an integral containing the current induced in the strip. Applying the boundary condition of vanishing tangential electric field along the strip leads to an integral equation by means of which the scattering cross-section is recast into the conventional variational form. To obtain an approximation to the true scattering cross-section, the authors use a constant current distribution. Although this current distribution violates the edge condition, it is stated that this simple assumption leads to remarkably accurate values of the cross-section over a wide range of frequencies, as is borne out by comparison with exact values of Morse and Rubenstein.

C. J. Bouwkamp (Eindhoven).

Pearson, J. D. The diffraction of electro-magnetic waves by a semi-infinite circular wave guide. Proc. Cambridge Philos. Soc. **49**, 659–667 (1953).

A plane electromagnetic wave is incident upon the mouth of a semi-infinite circular wave guide. It is shown that a direct appeal to the equations of Maxwell enables one to determine the longitudinal components of the electric and magnetic fields. This is done by essentially following a method discussed by D. S. Jones [Quart. J. Math., Oxford Ser. (2) **3**, 189–196 (1952); MR **14**, 474]. No numerical results are presented. *A. E. Heins* (Pittsburgh, Pa.).

Huang, Chaang, Kodis, Ralph D., and Levine, Harold. Diffraction by apertures. J. Appl. Phys. **26**, 151–165 (1955).

The general mathematical analysis of this paper is almost identical with that of Levine and Schwinger [Comm. Pure Appl. Math. **3**, 355–391 (1950); MR **13**, 305]. In the case of diffraction by a circular aperture, the variational formulation is applied to obtain an infinite set of linear equations for certain coefficients defining the field in the aperture. The case of the elliptic aperture is treated less satisfactorily in that a one-component trial field is assumed in the variational formulation, violating the edge condition. Despite this, measurements indicate that the agreement between experimental and theoretical results is good.

C. J. Bouwkamp (Eindhoven).

Meixner, Josef. Diffraction of electromagnetic waves by a slit in a conducting plane between different media. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-68**, i+11 pp. (1954).

The problem of diffraction of electromagnetic waves by an infinite slit in a perfect conductor at the interface between two different dielectric media is solved by the use of expansions of the fields in each medium in terms of the appropriate Mathieu functions. By matching these expansions across the slit, an infinite set of linear equations is obtained for the coefficients in the expansions of the diffracted fields. The solution is put in such a form that numerical results can be readily obtained. (Author's summary.)

E. T. Copson (St. Andrews).

Chambers, Ll. G. Diffraction by a half-plane. Proc. Edinburgh Math. Soc. (2) **10**, 92–99 (1954).

Following Friedlander [Proc. Roy. Soc. London. Ser. A. **186**, 322–344 (1946); MR **8**, 239] the author shows that the diffraction problem of the title for an arbitrary incident wave train, after introducing parabolic coordinates, leads to a certain integral equation. This equation can be readily solved when the incident wave is simply harmonic. The final expressions for the field components in this case are derived in full (but are not new). Sommerfeld's and Copson's solutions are stated to be complicated compared with the "elementary" solution of the author. [Really elementary solutions in terms of parabolic coordinates are those of H. Lamb [Hydrodynamics, 5th ed., Cambridge, 1924, p. 513] and J. Brillouin [C. R. Acad. Sci. Paris **229**, 513–514 (1949); MR **11**, 281], both of which are apparently unknown to the author. Copson is misquoted: ref. (2) should read: "On an integral equation arising in the theory of diffraction," Quart. J. Math. **17**, 19–34 (1946).]

C. J. Bouwkamp (Eindhoven).

Felsen, Leopold B. Backscattering from wide-angle and narrow-angle cones. J. Appl. Phys. **26**, 138–151 (1955).

The characteristic Green's function approach [cf. N. Marcuvitz, Comm. Pure Appl. Math. **4**, 263–315 (1951); MR **13**, 514] is employed to obtain alternative representations for the two-dimensional scalar Green's functions for a cone with Dirichlet or Neumann type boundary conditions. Convergence properties are studied in detail. Solutions are obtained for the diffraction of waves radiated by scalar or vector point sources on the axis of a semi-infinite cone. To evaluate the scattered wave at large distance from the cone tip, the author uses a contour integral representation and applies stationary phase methods [B. L. van der Waerden, Appl. Sci. Res. B **2**, 33–45 (1951); MR **12**, 808]. Backscattering from cones with large or small apex angles is calculated. Numerous references to similar work, much of which is not yet published, are given. *C. J. Bouwkamp*.

Hurd, R. A. The propagation of an electromagnetic wave along an infinite corrugated surface. Canad. J. Phys. **32**, 727–734 (1954).

An exact solution is given of a simplified problem first discussed by Brillouin [J. Appl. Phys. **19**, 1023–1041 (1948)]. On a perfectly conducting plate is built up a series of parallel, infinitely thin walls equal in height and evenly spaced. In the region above the slots a plane wave travels in the direction parallel to the plate and perpendicular to the slot walls, making the problem two-dimensional and scalar. An infinite system of equations is set up for certain physical parameters, which is solved by a method due to

Whitehead [Proc. Inst. Elec. Engrs. Part III. **98**, 133–140 (1951)]. Mode amplitudes and phase constant are evaluated and numerical results are discussed. *C. J. Bouwkamp.*

Furutsu, Koichi. On the theory of diffraction of the electromagnetic wave by mountains. J. Phys. Soc. Japan **8**, 500–524 (1953).

Solution of Maxwell's equations for the electromagnetic field taking into account the influence of mountains in the field, without assuming that the shape of the mountain is a very special one as for instance a sphere. It is assumed that the change of mountain surface characteristics, e.g. radii of curvature over a distance of a wavelength, is sufficiently small, and that those radii are large compared with the wavelength. The solution is given using Green's functions and the chief point of the paper is the construction and the investigation of those functions, given in vector-field notation throughout. They are considered first on or near the mountain surface and then at great distance from the mountain. *H. Bremerkamp* (Delft).

De Socio, Marialuisa. Sulla propagazione nelle guide con dielettrico eterogeneo. Riv. Mat. Univ. Parma **5**, 183–196 (1954).

In this paper is considered the general problem of the propagation of electromagnetic waves along a straight wave guide with perfectly conducting walls, the cross-section being simply connected but otherwise arbitrary. The guide is filled with a heterogeneous dielectric in which the dielectric constant is constant along lines parallel to the generators of the guide wall. It is assumed that this dielectric constant is of the form $\epsilon_0+\epsilon_1$, where ϵ_0 is constant but ϵ_1 is variable and very small compared with ϵ_0. Each mode of propagation is then a perturbation of a TE or TM mode with dielectric constant ϵ_0, and the usual perturbation techniques can be applied. *E. T. Copson* (St. Andrews).

Il'in, V. A. On the excitation of nonideal radio-wave guides. Dokl. Akad. Nauk SSSR (N.S.) **98**, 925–928 (1954). (Russian)

The theory of cylindrical wave-guides is extended in two directions, (i) the material of the guide is to be of good, but not ideal conductivity, (ii) the cross-sectional curve may have angular points. For (i) an approximate expression for the potential is found in terms of wave-functions for the ideal case, taken as far as the 1st or 2nd powers of k/k_0, the ratio of the wave numbers of the two media. For (ii), in the case of excitation by a magnetic dipole, there are supplementary terms of order $(k/k_0)^2$ having logarithmic singularities at the angular lines, which have a predominating influence near these lines. Approximate boundary conditions are used, as in a previous paper [same Dokl. **97**, 213–216 (1954); MR **16**, 652]. *F. V. Atkinson* (Oxford).

Il'in, V. A. Diffraction of electromagnetic waves from a nonideally conducting wedge and the problem of coastal refraction. Dokl. Akad. Nauk SSSR (N.S.) **99**, 47–50 (1954). (Russian)

The method of the previously-reviewed paper is applied to give an approximate expression for the magnetic potential due to a line source in the presence of such a wedge. An image method is then used to find another approximate solution in the case when the wedge angle is a submultiple of π. Qualitative deductions are then enunciated, without proof, regarding the effect on signal strength, for various polarisations, of the nearness of the transmitter to a coastal line. *F. V. Atkinson* (Oxford).

Miller, M. A. Application of homogeneous boundary conditions in the theory of thin antennas. Ž. Tehn. Fiz. **24**, 1483–1495 (1954). (Russian)

The author treats the problem of the cylindrical antenna, on whose surface the tangential electric and magnetic fields satisfy homogeneous boundary conditions: $E_z=Z_1H_\phi$, $E_\phi=-Z_2H_z$. The thickness of the antenna is assumed small compared to both the length of the antenna and the wavelength. With these assumptions the problem can be reduced to finding the magnetic and electric currents along the antenna. These satisfy separate equations, with Z_1 and Z_2 appearing in terms which are small for thin antennas. It is shown, however, that if Z_1 is allowed to vary with thickness, then the resonant frequency of the antenna may depend on Z_1 even in the limit of zero thickness. A few examples of practical interest are treated. *J. Shmoys.*

★Miller, M. A. Application of homogeneous boundary conditions in the theory of thin antennas. Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 24 pp. (mimeographed). $12.00.

Translation of the paper reviewed above.

Gaponov, A. V., and Levin, M. L. On the theory of thin antennas in resonant cavities. Dokl. Akad. Nauk SSSR (N.S.) **95**, 1193–1196 (1954). (Russian)

The authors treat the problem as oscillations on N antennas, coupled by the interaction due to their presence in the same cavity. The interaction is small when the antennas are thin and therefore a perturbation method is used. *J. Shmoys* (New York, N. Y.).

Ahiezer, A. I., and Lyubarskiĭ, G. Ya. On the theory of coupled resonant cavities. Ž. Tehn. Fiz. **24**, 1697–1706 (1954). (Russian)

The authors treat the problem of normal modes in two resonant cavities coupled by a narrow slit. Maxwell's equations in the space bounded by a perfect conductor are reduced to an integro-differential equation for the field in the slit, assuming that the ratio of length of the slit L to its width d is large. This equation is then solved by a perturbation method, in powers of α, where $\alpha^{-1}=\log(L/d)$. First-order effect on resonant frequencies is calculated. *J. Shmoys* (New York, N. Y.).

Kaganov, M. I., and Azbel', M. Ya. Conductivity of thin metallic films. Ž. Eksper. Teoret. Fiz. **27**, 762–763 (1954). (Russian)

The authors derive the effective conductivity of thin metallic sheets by considering the effect of external electric field on the distribution function. For the isotropic case the ratio of effective to bulk conductivity, expressed in terms of the sheet thickness d and mean free path of the electron l, turns out to be $(3d/4l)\log(l/d)$. *J. Shmoys.*

Epstein, Paul S. On the possibility of electromagnetic surface waves. Proc. Nat. Acad. Sci. U. S. A. **40**, 1158–1165 (1954).

The author considers the problem of existence of independent surface waves propagated along a plane separating two non-magnetic media. Such waves must have positive wave number and attenuation constant along the interface of the two media, must decay exponentially away from the interface, and must satisfy the radiation condition in the directions normal to, and away from the interface.

The author finds conditions under which waves satisfying all of the above, except the radiation condition, may exist. He concludes further that if the conductivities of both media are different from zero, no surface wave, satisfying all of the above conditions, can exist. If, however, both media are lossless, then surface waves can exist if one of the dielectric constants is positive, the other negative and if the sum of the dielectric constants is negative. *J. Shmoys.*

Marx, G., und Györgyi, G. Der Energie-Impuls-Tensor des Elektromagnetischen Feldes und die ponderomotorischen Kräfte in Dielektrika. Acta Phys. Acad. Sci. Hungar. **3**, 213–242 (1954). (Russian summary)

On the basis of consistent phenomenological description of the interaction of matter with electromagnetic fields (disregarding all molecular-structure aspects) it is shown that the total force upon the dipoles in the polarized dielectric may be expressed as a force needed to maintain the polarization against the quasielastic forces, and a Lorentz force acting upon the polarization current. The expression for the volume force-density agrees with the Abraham concept of the energy-momentum tensor [Theorie der Elektrizität, vol. II, 5th ed., Teubner, Leipzig, 1923] but not with that propounded by Minkowski. It is then shown that in crystalline dielectrics additional forces appear which produce a torque tending to rotate the crystal. This is again in agreement with experiment and with the expressions obtained from Abraham's stress tensor, whereas the Minkowski form leads to zero torque. Finally, it is demonstrated that the velocity of propagation of the radiation energy satisfies the Lorentz transformation. It is concluded that only the stress tensor of Abraham satisfies all requirements of experimental evidence and relativistic theory. *E. Weber.*

Györgyi, G. Die Bewegung des Energiemittelpunktes und der Energie-Impuls-Tensor des Elektromagnetischen Feldes in Dielektrika. Acta Phys. Acad. Sci. Hungar. **4**, 121–131 (1954). (Russian summary)

On the basis of the paper reviewed above, it is shown that the proper equations of motion for the center of energy or for the center of gravity of the dielectric exposed to an electromagnetic field are consistent with the energy-momentum tensor of M. Abraham. It is also demonstrated that other formulations of the energy-momentum tensor are not consistent with the requirements of experimental evidence. *E. Weber* (Brooklyn, N. Y.).

Lampariello, Giovanni. Considerazioni generali sulla propagazione delle onde elettromagnetiche nei corpi in moto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17**, 37–44 (1954). See the note on p. 1537

The author investigates the characteristics of the Maxwell-Minkowski equations in a medium moving with velocity v, with $(v/c)^2$ neglected, following the general method of T. Levi-Civita [Caratteristiche dei sistemi differenziali e propagazione ondosa, Zanichelli, Bologna, 1931]; he also discusses the propagation of plane waves, obtaining the usual Fresnel convection formula. [It appears to the reviewer that the author does not follow Levi-Civita correctly; the partial differential equation of the characteristics should be of the first order (like Hamilton's equation), and not, as shown in this paper, of the second order (like the wave equation). In spite of the author's assertion to the contrary, the Maxwell-Minkowski equations are derivable by a Lorentz transformation from the Maxwell equations for a fixed medium [cf. C. Møller, The theory of relativity, Oxford, 1952, p. 196; MR **14**, 212]; this fact reduces the interest of the paper.] *J. L. Synge* (Dublin).

Marziani, Marziano. Su un principio di minimo dell'elettrodinamica stazionaria dei superconduttori ciclici. Boll. Un. Mat. Ital. (3) **9**, 409–412 (1954).

In a superconductor in a steady state, the magnetic field $\mathbf{H}$ and the density $\mathbf{j}$ of supercurrent satisfy

$$(*)\qquad c \operatorname{rot} \mathbf{H}=\mathbf{j}, \quad c \operatorname{rot} \lambda\mathbf{j}=-\mathbf{H},$$

where c is a constant and λ a given function of position. The domain S of the superconductor here considered is $(n+1)$-ply connected, and σ_r $(r=1, 2, \cdots, n)$ is a set of ideal diaphragms which make S simply connected. The total currents are defined by

$$i_r(\mathbf{j})=\int_{\sigma_r} \mathbf{j}\times\mathbf{n}\, d\sigma_r,$$

where n is the unit normal to σ_r and $\times$ indicates the scalar product, and the total energy by

$$U(\mathbf{H}, \mathbf{j})=\frac{1}{2}\int_{S_\infty} \mathbf{H}^2\, dS_\infty+\frac{1}{2}\int_S (\mathbf{H}^2+\lambda\mathbf{j}^2)\, dS,$$

where S_∞ is all space outside S. The author compares $U(\mathbf{H}, \mathbf{j})$, where $\mathbf{H}, \mathbf{j}$ satisfy (*), with $U(\mathbf{H}_1, \mathbf{j}_1)$, where $\mathbf{H}_1, \mathbf{j}_1$ satisfy $c \operatorname{rot} \mathbf{H}_1=\mathbf{j}_1$ and $i_r(\mathbf{j}_1)=i_r(\mathbf{j})$ $(r=1, 2, \cdots, n)$, and proves that $U(\mathbf{H}, \mathbf{j})<U(\mathbf{H}_1, \mathbf{j}_1)$ unless $\mathbf{H}_1=\mathbf{H}$, $\mathbf{j}_1=\mathbf{j}$, thus establishing a minimum principle for total energy. A uniqueness theorem follows. *J. L. Synge* (Dublin).

Szymański, Z. Electrodynamics with higher derivatives as a particular case of electrodynamics with non-linear equations. Bull. Acad. Polon. Sci. Cl. III. **2**, 167–169 (1954).

The non-linear electrodynamics of Infeld and Plebański [Acta Phys. Polon. **12**, 123–134 (1953); MR **15**, 489] is made even more complicated by the, seemingly ad hoc, introduction of the Dalembertian into their definition of current. It then appears that the currentless case of the theory of Podolsky and Schwed [Rev. Mod. Phys. **20**, 40–50 (1948); MR **9**, 551] is a particular case of this new theory. The physical or mathematical interest of these speculations is neither apparent nor described.

A. J. Coleman (Toronto, Ont.).

Robin, M., et Rantz, J. Induction dans une veine liquide de section elliptique, circulaire ou rectangulaire. J. Rech. Centre Nat. Rech. Sci. no. **26**, 290–297 (1954).

The current density $\mathbf{j}$ in a fluid of electrical conductivity σ and magnetic permeability μ when a magnetic field $\mathbf{H}$ is impressed on the fluid is

$$\mathbf{j}=\sigma(-\operatorname{grad} U+\mu\mathbf{v}\times\mathbf{H}),$$

where U denotes the electric potential developed in virtue of a prevailing velocity field $\mathbf{v}$. If there are no additional sources of current, $\operatorname{div}\mathbf{j}=0$, and the differential equation governing U is

$$\nabla^2 U=\mu \operatorname{div}(\mathbf{v}\times\mathbf{H}).$$

The author considers the solution of this equation appropriate to a laminar flow of an incompressible fluid of constant σ and μ through an infinitely long cylinder of elliptic or rectangular cross-section when $\mathbf{H}$ is in the direction of the minor axis, in the elliptic case, and parallel to a side in the rectangular case. Thus, in the elliptic case the equation

to be solved is

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=4\mu\frac{HV_m}{a^2}x,$$

where V_m is the constant velocity along the axis of the cylinder and a is the major axis. A solution of this equation must be sought for which

$$\frac{x}{a^2}\frac{\partial U}{\partial x}+\frac{y}{b^2}\frac{\partial U}{\partial y}=0 \quad\text{on}\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The solution is found to be of the form

$$U(x,y)=2\mu HV_m\frac{(2a^2+b^2)x}{3a^2+b^2}\left[\frac{x^2}{3a^2}+\frac{y^2}{2a^2+b^2}-1\right].$$

A similar solution is found for the case of rectangular cross-section. *S. Chandrasekhar* (Williams Bay, Wis.).

Dungey, J. W. The motion of magnetic fields. Monthly Not. Roy. Astr. Soc. 113 (1953), 679–682 (1954).

One generally pictures the lines of magnetic force in a highly conducting medium as "frozen" in the medium. If, under these circumstances, the equations describing the lines of force are single valued then one can show that the growth or decay of a prevalent field occurs principally along neutral lines where the magnetic field is zero; and if the limiting lines of force near a neutral line are circular then ohmic losses are most pronounced here and decay of the existing field follows. In this paper the author points out that if the magnetic lines of force (instead of encircling) cross along the neutral lines then amplification of the existing field is possible by the lines of force breaking along the neutral lines and separating as two distinct lines of force.
S. Chandrasekhar (Williams Bay, Wis.).

Silverman, Richard A., and Balser, Martin. Statistics of electromagnetic radiation scattered by a turbulent medium. Phys. Rev. (2) 96, 560–563 (1954).

In an earlier paper by Villars and Weisskopf [Phys. Rev. (2) 94, 232–240 (1954); MR 15, 762] it was shown that the power of the radiant energy scattered by a medium in which the refractive index is subject to random fluctuation is given by

$$C(\mathbf{k},t)=\sum_{\mathbf{k}',\mathbf{k}''}(\mathbf{k}\cdot\mathbf{v}^t{}_{\mathbf{k}'})(\mathbf{k}\cdot\mathbf{v}^t{}_{\mathbf{k}-\mathbf{k}'})(\mathbf{k}\cdot\mathbf{v}^t{}_{-\mathbf{k}''})(\mathbf{k}\cdot\mathbf{v}^t{}_{\mathbf{k}''-\mathbf{k}}),$$

where $\mathbf{v}^t{}_{\mathbf{k}'}$ is the Fourier coefficient in the $\mathbf{k}'$ direction of the turbulent velocity field and $\mathbf{k}$ is the difference between the incident and the scattered propagation vectors. On the assumption

$$\langle\mathbf{v}^t{}_{\mathbf{k}'}\cdot\mathbf{v}^t{}_{-\mathbf{k}'+\mathbf{k}''}\rangle_{Av}=\delta_{0\mathbf{k}''}\langle|\mathbf{v}^t{}_{\mathbf{k}'}|^2\rangle_{Av},$$

the authors show that the nth moment of the distribution of $C(\mathbf{k},t)$ is given by

$$\langle C^n(\mathbf{k},t)\rangle_{Av}=n!\langle C(\mathbf{k},t)\rangle_{Av}{}^n$$

and that therefore it is Gaussian.

Next, considering the joint probability distribution of $\xi=C(\mathbf{k},t)$ and $\eta=C(\mathbf{k},t+\tau)$, the authors show on the assumption

$$\langle\mathbf{v}^t{}_{\mathbf{k}'}\cdot\mathbf{v}^{t+\tau}_{-\mathbf{k}'+\mathbf{k}''}\rangle_{Av}=\delta_{0\mathbf{k}''}\langle\mathbf{v}^t{}_{\mathbf{k}'}\cdot\mathbf{v}^{t+\tau}_{-\mathbf{k}'}\rangle_{Av}$$

that

$$\langle\xi^r\eta^s\rangle_{Av}=2^{r+s}r!s!\sum_{l=0}^{\min(r,s)}\binom{r}{l}\binom{s}{l}\sigma_0{}^{r+s-2l}\sigma_\tau{}^{2l}$$

where

$$\sigma_0=\sum_{\mathbf{k}'}\langle|\mathbf{k}\cdot\mathbf{v}^t{}_{\mathbf{k}'}|^2\rangle_{Av}\langle|\mathbf{k}\cdot\mathbf{v}^t{}_{\mathbf{k}-\mathbf{k}'}|^2\rangle_{Av}$$

and

$$\sigma_\tau=\sum_{\mathbf{k}'}\langle(\mathbf{k}\cdot\mathbf{v}^t{}_{\mathbf{k}'})(\mathbf{k}\cdot\mathbf{v}^{t+\tau}_{-\mathbf{k}'})\rangle_{Av}\langle(\mathbf{k}\cdot\mathbf{v}^t{}_{\mathbf{k}-\mathbf{k}'})(\mathbf{k}\cdot\mathbf{v}^{t+\tau}_{\mathbf{k}'-\mathbf{k}})\rangle_{Av}.$$

From these formulae it follows in particular that

$$W(\xi,\eta)=\frac{1}{4\sigma_0{}^2(1-\rho^2)}e^{-(\xi'+\eta')}I_0(2\rho\sqrt{(\xi'\eta')})$$

where $I_0(x)$ is the Bessel function of order zero and imaginary argument, and $\rho=\sigma_\tau/\sigma_0$ and

$$\xi'=\xi/2\sigma_0(1-\rho^2),\quad \eta'=\eta/2\sigma_0(1-\rho^2).$$

S. Chandrasekhar (Williams Bay, Wis.).

Ştefănescu, Sabba S. Le champ magnétique des courants électriques stationnaires dans les milieux hétérogènes a. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 5, 199–206 (1953). (Romanian. Russian and French summaries)

Cade, R. An electrostatic problem involving a non-linear fluid dielectric. Proc. Phys. Soc. Sect. B. 68, 1–9 (1955).

The author chooses an anisotropic solid dielectric sphere of radius a immersed in a fluid whose dielectric constant is slightly field-dependent, namely $\epsilon=\epsilon_0(1+\nu E^2)$ if E is the field vector. In a previous paper [same Proc. 66, 557–569 (1953); 67, 689–704 (1954); MR 15, 185] the solution for the linear case $\nu=0$ is given. Applying a perturbation method, the author determines the first correction term for the potential in the nonlinear case. Outside the sphere, a double series of associated Legendre functions results, inside the sphere the author uses superposition of simple solutions of the potential equation in ellipsoidal coordinates obtained by transforming the cartesian coordinates to $\xi=\epsilon_x{}^{-1/2}x$, $\eta=\epsilon_y{}^{-1/2}y$, $\zeta=\epsilon_z{}^{-1/2}z$, where ϵ_x, ϵ_y, ϵ_z are the dielectric constants in the respective directions. The uniqueness of the solution is linked to the convergence of the perturbation method but is not proved.

Based on the stress tensor developed by G. H. Livens [Theory of electricity, 2d ed., Cambridge, 1926], the electrostatic couple is computed and analogies to paramagnetic and steady current problems are stated. Measurements are proposed to test the influence of polar liquids upon the electrostatic couple and thus determine ν above. However, the effect of a double layer, which has been excluded in the boundary conditions, could be of considerable significance.
E. Weber (Brooklyn, N. Y.).

Shekel, Jacob. Indefinite admittance representation of linear network elements. Bull. Res. Council Israel 3, 390–394 (1954).

The analysis of circuits constructed with linear elements having more than two terminals is considered. The n terminals of the element may be partitioned into s subsets. The following two postulates are made: (i) the sum of all the currents entering the terminals of any subset is zero for any arbitrary set of applied currents; (ii) an arbitrary voltage added to the voltages of all the terminals of any subset has no effect on the current into any terminal. As a consequence of these two postulates an admittance matrix relating the currents and voltages, which is square and of order n, has the special structure such that when it is partitioned both horizontally and vertically according to the s subsets, then each of the resulting s^2 submatrices has the property that the sum of the elements in each row and column is zero.
R. Kahal (Syracuse, N. Y.).

Zemanian, A. H. Further bounds existing on the transient responses of various types of networks. Proc. I. R. E. 43, 322–326 (1955).

Relations between the transient response of a network and its frequency system function are treated. Several theorems are derived dealing with the bounds on the transient response of various types of networks when the system function is suitably restricted. Thus, for example, if the real part of the system function is a monotonically decreasing function of the real frequency, the step response of the network cannot have an overshoot of more than eighteen percent of its final value. Moreover, a lower bound on the rise time can be expressed in terms of certain coefficients of the power series and inverse power series expansions of the system function. Other kinds of system function restrictions lead to expressions for other transient response bounds.

R. Kahal (Syracuse, N. Y.).

Ruhrmann, Alfred. Symmetrische Vierpole bei transformierendem Betrieb. Arch. Elektrotech. 41, 320–333 (1954).

In analogy to the exponential transmission line the theory of symmetrical four-pole chains is developed for asymmetrical operation whereby the fourpole is characterized by its transforming properties. The concepts of transforming characteristic and iterative impedances and transforming propagation constant are established and related to pass bands in frequency similar to, but different from the conventional four-pole quantities. Current and voltage transformer ratios are introduced, a circle diagram developed, and an illustrating example given. The relations of this new branch of four-pole theory permit the systematic design of the equivalent to continuous exponential lines.

E. Weber (Brooklyn, N. Y.).

Quantum Mechanics

Landé, Alfred. Quantum indeterminacy, a consequence of cause-effect continuity. Dialectica 8, 199–209 (1954).

***Einstein, Albert. Einleitende Bemerkungen über Grundbegriffe. Remarques préliminaires sur les concepts fondamentaux.** Louis de Broglie, physicien et penseur, pp. 4–15. Editions Albin Michel, Paris, 1953. 870 francs. (French and German)

In this short article Einstein states his objections to the usual statistical interpretation of the wave functions of quantum mechanics. His main thesis is that the description of the states of a physical system by means of the wave functions is incomplete. He states his belief that a complete physical theory will describe "real states" which are independent of observations or measurements. *A. H. Taub.*

Destouches, Jean-Louis. Sur l'interprétation physique de la mécanique ondulatoire et l'hypothèse des paramètres cachés. J. Phys. Radium (8) 13, 354–358 (1952).

The process of measurement is analyzed in terms of a "hidden-parameter" interpretation of quantum mechanics. It is shown that a consistent analysis in these terms is always possible for a measurement carried out on a prescribed system with prescribed apparatus, but the descriptions of different measurements carried out on the same system will not always be consistent. *F. J. Dyson.*

Destouches, Jean-Louis. Sur l'interprétation physique des théories quantiques. J. Phys. Radium (8) 13, 385–391 (1952).

This is a semi-metaphysical discussion of the meaning of quantum mechanics, following the ideas of the paper reviewed above. The second section is entitled "Quanta and cosmology" and consists of speculation about the relation of quantum mechanics to a unified theory of the universe on a cosmological scale. *F. J. Dyson.*

Petiau, Gérard. Sur la détermination des fonctions d'ondes à singularités localisées de la théorie de la double solution dans quelques cas de conditions aux limites simples. C. R. Acad. Sci. Paris 240, 848–850 (1955).

Takabayasi, Takehiko. Remarks on the formulation of quantum mechanics with classical pictures and on relations between linear scalar fields and hydrodynamical fields. Progr. Theoret. Phys. 9, 187–222 (1953).

This is a sequel to the author's earlier paper [same journal 8, 143–182 (1952); MR 14, 705]. Many topics are discussed, all relating to the problem of interpreting quantum mechanics by means of an ensemble of classical motions. Limitations and inconsistencies in the ensemble picture are demonstrated. The main novelty in this paper is a discussion of the quantum mechanics of a relativistic particle obeying the Klein-Gordon equation and interacting with the electromagnetic field. The classical ensemble picture for such a particle leads to equations identical with the "new classical electrodynamics" of P. A. M. Dirac [Proc. Roy. Soc. London. Ser. A. 212, 330–339 (1952); MR 14, 228]. Analogies are also found between the ensemble equations and the equations of hydrodynamics. The discussion is broken off before any physical consequence of the analogy becomes apparent. *F. J. Dyson* (Princeton, N. J.).

Suffczyński, Maciej. Quantization of non-linear electrodynamics. Acta Phys. Polon. 13, 291–299 (1954). (Russian summary)

The method of P. A. M. Dirac [Canad. J. Math. 3, 1–23 (1951); MR 13, 306], for setting up the dynamical equations of a system in which the canonical momenta are not independent, is here applied to the non-linear electrodynamics of M. Born and L. Infeld [Proc. Roy. Soc. London. Ser. A. 150, 141–166 (1935)]. The Hamiltonian of the theory is determined explicitly by this method, and the formal quantization can then be completed in the usual way.

F. J. Dyson (Princeton, N. J.).

Ortiz Fornaguera, R. Functional analysis in relation to the Dirac formalism for localizable dynamical systems. Rev. Acad. Ci. Madrid 46, 315–346 (1952). (Spanish)

The paper is devoted to a very detailed but elementary discussion of the use of space-like surfaces in quantum theory of fields, mainly in the line of the general paper of Dirac on the subject [Phys. Rev. (2) 73, 1092–1103 (1948); MR 10, 225]. *L. Van Hove* (Utrecht).

Ortiz Fornaguera, R. On the variance of the quantities in the canonical formalism. Rev. Acad. Ci. Madrid 46, 137–156 (1952). (Spanish)

The author discusses the Lagrange and Hamilton formalisms for a classical field theory, paying special attention to the variance of the various quantities involved with respect to arbitrary transformations of the independent variables, as well as of the parameters of the hypersurface S enclosing

the region over which the action integral is taken. It is shown that the canonical formalism can be carried through without the assumption of a metric or an affine structure for the space of independent variables. It requires only the choice in each point of S of a contravariant vector not tangent to S. *L. Van Hove* (Utrecht).

Serpe, J. Remarques sur l'application de la théorie des distributions à la théorie quantique des champs. Physica **20**, 736–742 (1954).

This is a criticism of the paper of W. Güttinger [Phys. Rev. (2) **89**, 1004–1019 (1953); MR **15**, 85] who used the theory of distributions to obtain finite values for the divergent integrals of quantum electrodynamics. The author states that the distribution theory by itself leads to precisely the same divergent expressions as the ordinary methods of calculation, and that Güttinger's results were obtained by "an abuse of the concept of pseudo-function." The reason for this disagreement is that the divergences arise from multiplying together singular functions; the distribution theory does not in general give a meaning to the (ordinary) product of two distributions; therefore the definition of the product of two singular functions remains ambiguous when the functions are replaced by distributions.
F. J. Dyson (Princeton, N. J.).

Novožilov, Yu. V. On the quantum theory of a field with causal operators. Dokl. Akad. Nauk SSSR (N.S.) **99**, 723–726 (1954). (Russian)

Continuing the analysis of his earlier paper [same Dokl. (N.S.) **99**, 533–536 (1954); MR **16**, 546], the author now deduces the commutation laws for causal operators at any two space-time points. He shows that the equation of motion in the interaction representation is automatically integrable, and leads to the correct perturbation-theory expansion of the S-matrix. It now appears that the author's formalism is identical with that of F. Coester [Phys. Rev. (2) **89**, 619–620 (1953); MR **14**, 828] although developed independently. A knowledge of Coester's work makes the meaning of these papers much clearer. *F. J. Dyson* (Princeton, N. J.).

Symanzik, Kurt. Über das Schwingersche Funktional in der Feldtheorie. Z. Naturf. **9a**, 809–824 (1954).

This paper gives a clear and useful survey of the formal relations between the functional methods of R. P. Feynman [Phys. Rev. (2) **84**, 108–128 (1951); MR **13**, 410] and of J. Schwinger [Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951); MR **13**, 520] in quantum field theory. It is shown how the functionals may be expanded in Volterra series, the coefficients being then ordinary functions in configuration space, called Green's functions by Schwinger. Various approximate methods of examining these functions are described, in particular, the method called "New Tamm-Dancoff method" by the reviewer [Phys. Rev. (2) **91**, 1543–1550 (1953); MR **15**, 768]. The main new result in this paper is a proof that the NTD method fails completely when applied to the simple and well-understood example of a one-dimensional anharmonic oscillator. It is therefore very unlikely that the method will be of any use for the much more difficult problems of field theory. *F. J. Dyson.*

Bogolyubov, N. N. On representation of Green-Schwinger functions by means of functional integrals. Dokl. Akad. Nauk SSSR (N.S.) **99**, 225–226 (1954). (Russian)

It is shown that functional integrals of the kind introduced by R. P. Feynman [Phys. Rev. (2) **80**, 440–457 (1950); MR **12**, 889] constitute a formal solution of the equations of a quantized field theory. The content of this paper is almost a repetition of that of E. S. Fradkin [same Dokl. (N.S.) **98**, 47–50 (1954); MR **16**, 317]. *F. J. Dyson.*

Silin, V. P. On a modified Tamm method. Ž. Eksper. Teoret. Fiz. **27**, 754–756 (1954). (Russian)

Using the method of Tamm and Dancoff [i.e., Tamm, Acad. Sci. USSR. J. Phys. **9**, 449–460 (1945)] the author sets up the scattering equation for one meson and one proton in the lowest approximation. It is shown how the proton and meson self-energies may be consistently renormalized, following the ideas of M. Cini [Nuovo Cimento (9) **10**, 614–629 (1953); MR **15**, 768]. *F. J. Dyson* (Princeton, N. J.).

Freistadt, Hans. Classical field theory in the Hamilton-Jacobi formalism. Phys. Rev. (2) **97**, 1158–1161 (1955).

The formal construction of a Hamilton-Jacobi equation is carried through for the case of classical relativistic field dynamics. A time-independent and a time-dependent formulation are outlined, and the relation between them is examined. The methods are illustrated by means of the free Dirac and Klein-Gordon field-equations, for which the solutions are obtained explicitly. *F. J. Dyson.*

Yamazaki, Kazuo. Operator calculus in quantized field theory. Progr. Theoret. Phys. **7**, 449–468 (1952).

This is a formal analysis of the S-matrix and the Green's functions of quantum field theory, using the functional integral technique of R. P. Feynman [Phys. Rev. (2) **84**, 108–128 (1951); MR **13**, 410]. In particular, modifications of Feynman's method are introduced which make the method apply formally both to commuting and to anti-commuting fields. As an example, the formulas for the multiple production of bosons or fermions by a given external source-function are calculated explicitly.
F. J. Dyson (Princeton, N. J.).

Fujiwara, Izuru. Operator calculus of quantized operator. Progr. Theoret. Phys. **7**, 433–448 (1952).

This is mainly an expository account of the operator calculus of R. P. Feynman [Phys. Rev. (2) **84**, 108–128 (1951); MR **13**, 410], with a detailed application of the method to the dynamics of a forced harmonic oscillator.
F. J. Dyson (Princeton, N. J.).

Thermodynamics, Statistical Mechanics

Wundheiler, Alexander W. Irreversible systems, entropy, and Riemann spaces. Proc. Nat. Acad. Sci. U. S. A. **40**, 844–848 (1954).

In Part 1 of this paper the author develops certain fundamental thermodynamic concepts in a very brief but (in the reviewer's opinion) artificial and implausible manner, using the quantity frigor (reciprocal temperature). In Part 2 the concept of a reversible change is generalized: a (possibly discontinuous) change ab of a system from state a to state b (they may be nonequilibrium states) is "regular" if both ab and ba are producible, so that the residual complementary changes Γ and Γ' occur in reservoirs only, Γ and Γ' being independent of each other. The lower bound of the entropy change $\sigma(\Gamma)$ in Γ is called the "coentropy" change, σab,

in ab. Let

$$2Rab=\sigma ba+\sigma ab,\quad 2Sab=\sigma ba-\sigma ab.$$

Properties of the degradation Rab and the pre-entropy Sab are developed. The Clausius entropy theorem is proved without the usual restrictions, and a condition is given for the entropy of a system to exist. In Part 3 these ideas are further developed for infinitesimal displacements, and the coefficients in the quadratic form dR^2 are identified. Conditions are given under which the usual thermodynamic quantities, such as internal energy, may be defined, and the significance of Onsager's coefficients is clarified.

C. C. Torrance (Monterey, Calif.).

Landé, Alfred. Thermodynamic continuity and quantum principles. Phys. Rev. (2) **87**, 267–271 (1952).

The "Gibbs paradox" in thermodynamics is the fact that the diffusional mixing of two volumes of gas produces an increase of entropy if the gases are composed of atoms of different species, while there is no increase of entropy if the atoms are alike. The paradox appears because in classical physics one can imagine the atoms being made "gradually" more and more alike, and there will be a discontinuity in the entropy when they become identical. The paradox is resolved in quantum physics because the likeness of two atoms is then a yes-or-no question and cannot be varied continuously even in imagination. The author here develops a chain of semi-logical argument by which the principles of quantum statistics are deduced from the non-existence of a Gibbs paradox. The quantum phenomenon of the "degree of overlap" of two states of a system is defined in terms of the reversibility or irreversibility of mixing. The author claims that these results "might have been found by Gibbs through pure reasoning concerning his paradox." *F. J. Dyson.*

Lax, Melvin. Relation between canonical and micro-canonical ensembles. Phys. Rev. (2) **97**, 1419–1420 (1955).

Klein, Martin J. Principle of detailed balance. Phys. Rev. (2) **97**, 1446–1447 (1955).

Katsura, Shigetoshi. On the theory of cooperative phenomena. Progr. Theoret. Phys. **11**, 476–492 (1954).

The author considers the Ising model for relatively small models in the toroidal form and subject to a magnetic field. The spinor notation is used to obtain the partition function for (2×2), (2×3) and (3×3) plane lattice and the $(2\times2\times2)$ and $(3\times3\times2)$ cubic lattice. The author indicates the results of a numerical investigation on the finite equivalent of the Onsager-Kaufman procedure. The author also compares the free energy, the internal energy and the specific heat for these lattices with the corresponding results of Onsager in two dimensions and Wakefield in the three-dimensional case. The author also uses these models to discuss the lattice theory for condensation and establishes the two-phase phenomena even in the small models. *F. J. Murray.*

***Hauptman, Herbert, and Karle, Jerome. Solution of the phase problem. I. The centrosymmetric crystal.** ACA Monograph No. 3. American Crystallographic Association, 1953. 87 pp. $1.50 (copies can be obtained from The Letter Shop, Inc., 222 West 8th St., Wilmington 1, Delaware).

In the determination of the atomic structure of crystals use is made of the diffraction patterns which are produced when a monochromatic beam of X-rays irradiates a crystalline specimen, because the diffraction maxima associated with the various families of planes determined by the periodic atomic structure are proportional to the Fourier coefficients in the expansion of the electron density distribution. In the process of observing the diffraction maxima, however, the phase relationships are lost and only the absolute magnitudes of the reflections are recorded. As a result of this, structure synthesis is an art that requires many types of information to supplement that furnished through X-ray evidence alone.

The authors of this monograph have attacked the problem of phase determination by developing a statistical theory that yields a probability that the phase of a given Fourier coefficient lie within specified limits; e.g. that a real coefficient be positive. Since, generally, there are more structure factor amplitudes (diffraction maximum amplitudes) observed than the $3N$ coordinates of the N atoms in the unit cell, the authors use this abundant information to obtain probabilities close to certainty and thus claim to have solved the phase problem.

In Chapter I, the statistical theory is developed on the assumption that the atoms are independently and uniformly distributed within the unit cell. The atomic coordinates are taken as the independent random variables, each uniformly distributed in the interval $(0, 1)$. The characteristic functions of the probability distributions of the structure factors are obtained by making use of the known relations, for the various types of crystal symmetries, between structure factors and atomic scattering factors.

In Chapter II, the four types of centrosymmetric groups are discussed in detail. This chapter also includes a discussion of the equivalent co-ordinate origins, since the phase of a structure factor depends on the origin to which the expansion is referred. The permissible origins are those that maintain the class symmetry. Certain linear phase combinations are investigated which are independent of the choice of permissible origin. These are called phase invariants. Other linear phase combinations independent of the choice of origin permitted by a chosen form of the structure factor are called phase seminvariants. This chapter is somewhat parenthetical.

Chapter III deals with the construction of the joint distributions of groups of related structure factors and conditional distributions for individual structure factors in these groups. No criterion is presented for selecting the conditional distribution that yields the optimum probability for the sign of a structure factor.

The application of the theory to structure determination is presented in detail in Chapter IV. At the beginning of the analysis, before any signs are known, the conditional probability distribution of a given structure factor must be an even function of the other structure factors involved. This restriction is gradually removed as signs are determined.

A modification of the theory, required when the crystal contains atoms that occupy special positions, is developed in Chapter V.

The assumption of independence of atomic coordinates has been forced on the authors by the analytical complexity of a more realistic model. In the real crystal, of course, each atom excludes the others from the volume it occupies. Aside from this the authors present no justification for considering the atomic coordinates (or any other parameters, for that matter) as random variables in the accepted statistical

sense. Consequently, it is difficult to assign a meaning to the probabilities obtained in this analysis. In the construction of joint distributions it is not indicated how to avoid obtaining degenerate distributions. The authors have not taken advantage of some of the well-known properties of characteristic functions and the corresponding distributions and as a consequence their analysis appears cumbersome. By failing to present the conditional distributions in terms of the standard variable $z=(x-m)/\sigma$, where m, σ are the conditional mean and standard deviation, the working formulas lack compactness and clarity. In fact, instead of using probabilities the authors are obliged to use certain sums that are monotonically related to the probabilities. *J. J. Slade.*

***Vand, Vladimir, and Pepinsky, Ray. The statistical approach to X-ray structure analysis.** X-Ray and Crystal Analysis Laboratory, Department of Physics, The Pennsylvania State University, State College, Pa., 1953. xvi+98 pp.

This monograph has been written to refute the claim of Hauptman and Karl [see the book reviewed above] to have solved the phase problem and also to develop what the authors consider to be the useful part of the statistical theory of phase determination. No systematic plan of development is presented and the misstatements that occur throughout the text tend to destroy the force of their criticism. *J. J. Slade* (New Brunswick, N. J.).

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***Mathematics in type.** The William Byrd Press, Richmond, Va., 1954. xii+58 pp. $3.00.

Discussion of mathematical composition on monotype machines and its effect on mathematical style. Chapter headings: Factors affecting difficulty of composition; Methods of composition; Setting and spacing; The eccentricities of letters; Preparing and marking the manuscript; Variables of style in manuscript, in type; Proof changes and corrections; Kinds and sizes of types.

***Herland, Leo. Dictionary of mathematical sciences. Vol. II. English-German.** Frederick Ungar Publishing Co., New York, 1954. 336 pp. $4.50.

For vol. I [Ungar, New York, 1951] see these Rev. **13**, 418.

Jahrestagung der Gesellschaft für angewandte Mathematik und Mechanik, 1954.

The abstracts of the papers presented at this Jahrestagung are printed in Z. Angew. Math. Mech. **34**, Heft 8/9 (1954). They will not be reviewed separately.

Tolfte Skandinavska Matematikerkongressen i Lund, 10–15 Augusti 1953.

This publication carries the subtitle, Comptes Rendus du Douzième Congrès des Mathématiciens Scandinaves tenu à Lund, 10–15 août 1953. It is a volume of xvi+337 pp., was published in Lund in 1954, and may be purchased for 25 Swedish crowns from Lunds Universitets Matematiska Institution, Lund, Sweden.

Acta Mathematica Sinica.

This publication of the Academia Sinica started with vol. 3, no 1, in 1953. It succeeds the Journal of the Chinese Mathematical Society (New Series) (see below). Papers are in Chinese with a summary in a European language.

Comunicările Academiei Republicii Populare Romîne.

Vol. 1 of this journal appeared in 1951. Volumes 2, 3, and 4 appeared in the succeeding years. Papers are in Romanian with French and Russian summaries.

International Journal of Abstracts on Statistical Methods in Industry. Bibliographie Internationale Annotée des Applications de la Statistique.

Vol. 1, no. 1, of this periodical is dated 1954. Reviews are in English or French and are grouped under the following classifications: process control; sampling and acceptance inspection; research and development; administrative applications and operations research; management applications. The present plan is to publish three issues a year, each of about 100 abstracts. The abstracts are printed on card board on only one side, and are bordered so that they may be cut out and filed as 3″×5″ cards. Subscriptions may be entered with the publisher: International Statistical Institute, 2 Oostduinlaan, The Hague.

Journal of the Chinese Mathematical Society.

A new series of this journal started with Vol. 1, no. 1, dated March 1951. It continued for two volumes and was succeeded by Acta Mathematica Sinica (see above).

Prace Matematyczne.

Vol. 1, no. 1, of this new publication of the Polskie Towarzystwo Matematyczne is dated 1955. Papers are in Polish.

Studia Logica.

Vol. I of this publication Polish Academy of Sciences is dated 1953. Vol. II appeared in 1955. Papers written in Polish have summaries in Russian and in English, French, or German.

Vyčislitel'naya Matematika i Vyčislitel'naya Tehnika.

The first issue (vypusk) of this publication is dated 1953. It is a publication of the Institut Točnoĭ Mehaniki i Vyčislitel'noĭ Tehniki of the Akademiya Nauk SSSR.

Wiadomości Matematyczne.

Vol. 1, no. 1, of the second series of this journal is dated 1955. Papers are in Polish and are chiefly of an expository nature. It is a publication of the Polskie Towarzystwo Matematyczne.

Zeitschrift für mathematische Logik und Grundlagen der Mathematik.

Band 1, Heft 1, of this journal is dated 1955. It is edited by G. Asser and K. Schröter of the Institut für Mathematische Logik der Humboldt-Universität, Berlin, and is published by the Deutscher Verlag der Wissenschaften, Berlin.

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